ELASTIC SOIL – STRUCTURE INTERACTION

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Synopsis

This paper presents a refined finite element analysis for the analysis of two-dimensional plane stress and plane strain structures with particular emphasis being placed on the ability to solve problems of soil-structure interaction under earthquake loadings. The structure and the soil are idealized as an assemblage of quadrilateral plane stress and plane strain elements having a cubic variation in displacement enabling a more accurate representation of the stiffness properties of the system than that previously available. The response of the system to the earthquake acceleration history is achieved by a superposition of normal mode responses and the methods of obtaining the mode shapes and frequencies are outlined. Examples are presented to illustrate the capability of this approach.

Introduction

The most common method for obtaining the earthquake response of a structure to an earthquake accelerogram is to assume that the soil under the structure is rigid relative to the structure and therefore apply the accelerogram as input to the base of the structure. For many structures, particularly those founded on hard foundations, or where little interaction between the ground and the structure is expected, this method will be quite satisfactory in giving accurate indications of the deflections of the structure and the forces in its members.

However, there are many instances where the effect of the ground flexibility will have a marked effect on the behaviour of the structure under free vibration, i.e. the natural periods and mode shapes are altered by the ground motion. Also the layers of relatively soft soil overlying the relatively rigid bedrock will have its own dynamic characteristics and hence will modify the input accelerations arriving at the bedrock and may amplify or diminish the amplitude at the ground surface and will also vary the frequency content of the surface accelerogram.

If the soil layers are of large extent and depth relative to the structure then it could be possible, with little error, to account for these effects separately. This has been attempted by Shepherd and Donald[1] and Palmer [2] who accounted for the flexibility of the ground and its effects on the natural modal characteristics of the structure by introducing fictitious rotational and translational springs at the base of the structure. However to account for the flexibility in this way requires a reasonable amount of experience in allocating the spring constants to these artificial springs. The effect of the soil layers on the accelerogram has also received a considerable amount of investigation[3,4,5] by use of the shear beam or shear building analogy. In this approach the layers and sub-layers are assumed to deform in shear only and to be horizontal, uniformly thick and infinite in extent in the horizontal direction. By this method of analysis, which is identical to that of a shear building, it is possible to derive, from the accelerogram at bedrock, a record at the ground surface which can then be used to analyse the structure using this derived record as the base input to the structure.

Such approaches are only of any real value if the extent of the soil layers are such that the structure will have a negligible effect on the behaviour of the soil and if the width of the soil system is such that it is large in horizontal extent compared with its depth and that the structure is located away from the edges of this soil system. If this cannot be realised then the only way to analyse the structure and achieve an accurate solution is to use a form of analysis where the soil and the structure can be analysed as a combined system.

As the soil system is a continuous body the equation relating forces and displacement is a partial differential equation in x and y (the two plane co-ordinates of the vertical plane considered in the two dimensional analysis) and its formulation may be very difficult, if not almost impossible, if the elastic properties of the soil vary with position and the boundaries of the soil layers are not regular. To overcome these difficulties a numerical method is to be preferred, and though not leading to a closed form solution and having to be solved for each particular structure and site, this is of little consequence today with the widespread availability of the digital computer.

The most suitable method for the analysis of such system is the recently developed finite element method which has considerable success in the static and dynamic analyses of continuous stuctures [6,7,8]. It has also received some attention in soil-structure interaction [9,10] as well as in the analysis of earthdams and the effect on earth embankments.

The Finite Element Method

In the finite element method the continuum is replaced by an assemblage of regions or

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elements connected together at a finite number of points or nodal points. The shapes of the elements are usually triangular, rectangular or quadrilateral in the analysis of two dimensional systems and they may have nodal points along the edges of the elements as well as at the corners of the elements. The advantage of the triangle and the quadrilateral with respect to the rectangle is that irregular boundaries can be readily accommodated in a much more accurate manner than that possible with the rectangular elements.

Every finite element has a displacement field which is constrained to belong to a finite class of functions, each of which is continuous within the region of the element and which satisfies continuity of deformations across the interfaces between adjoining elements. The co-ordinate functions are known as the displacement shapes and they are uniquely defined by the deformation of the nodal points on the element which can be regarded as kinematic degrees of freedom. This provides a displacement field, for the connected, discretized continuum, that is continuous, piece-wise differentiable and can be made to satisfy any kinematic boundary conditions.

The result is that the equations of equilibrium of each of the nodal points which can be derived using strain energy or virtual work approaches, represent a lumped parameter representation of the stiffness properties of the continuum. As there are a finite number of nodal points and a finite number of equations of equilibrium for each nodal point there are now a finite number of kinematic degrees of freedom for this discretized continuum compared with the infinite number of the real continuum. The finite element has been shown, mathematically, to converge to the true solution as the number of elements is increased and therefore any desired accuracy, within the limits of the available computer storage limits and word length, can be achieved. However, it is almost impossible to state "a priori" how accurate a given analysis is, and so it is quite common to repeat the analysis, increasing or decreasing the number of elements and comparing the two solutions.

The advantages of the method are that the lumped parameter formulation permits the partial differential equation in x, y and time t to be replaced by a series of simultaneous ordinary differential equations varying in time only. Also, the discrete formulation of each finite element permits a great complexity of material properties to be permitted and thus the soil layers need not be infinite in extent or have horizontal interfaces and as well the geometry of the bedrock-soil boundary may be irregular as may the soil surface profile.

Furthermore various types of finite elements may be combined within a single structure, so that it is possible to include plane stress and plane strain elements together, to idealize the structure and the soil and provided care in the programming of the different types of degrees of freedom is observed beam members can also be incorporated within the analysis[10].

The finite element analyses of soil-structure problems to date have used relatively simple finite elements in that the displacement fields permitted within each element are simple polynomial expressions such as linear or quadratic polynomials in x and y. Such elements are the 'constant strain triangle' the 'linear strain triangle' and the 'linear strain rectangle'. All these elements have only the x and y displacement at each nodal point and the total number of kinematic degrees of freedom is two times the number of nodal points.

These simple elements have the disadvantage that in order to accommodate the high strain gradients that may occur in some regions of the structure a very fine mesh element idealization may be required. This may then be achieved in one of two ways. Either a fine mesh is drawn up as the original data, this increases the work required in preparing the data and increases the chance of data errors. Furthermore, as the number of nodes increases the number of masses will increase and hence the number of dynamic degrees of freedom will increase and hence the size of the eigenvalue problem to be solved for the mode shapes and frequencies will increase. The available computer will undoubtably limit the size of the mesh permitted and hence the mesh refinement with the result that the stiffness properties of the continuum are poorly represented.

As the mass distribution within the continuum, however, does not require such a fine nodal point distribution, a second alternative is to subdivide each finite element into smaller finite elements. This can be done internally within the computer program and hence will require no extra data preparation. The actual continuum then becomes an assemblage of "super-elements" each one of which is a finite element structure of its own. If the internal nodes of this element do not have masses associated with them, i.e. all the mass of the super-element is associated with the boundary nodes of the element then the internal kinematic degrees of freedom may be eliminated by an inverse Gauss elimination method.

However, the nodes along the sides of the element are a nuisance in the direct stiffness formulation of the total stiffness matrix of the structure, lead to inefficiency in the total formulation and increase the chance of data errors. As only corner nodes are preferred on the finite elements the nodes along the sides can be constrained to have a linear displacement variation with respect to the corner nodes. The nett result is an element with much greater internal flexibility though still with the disadvantage of a constrained strain field along the element edges.

A similar effect is noted with the finite elements used by Wilson[9] and Khanna[10]. In the element used by Wilson the element interior has a quadratic variation of displacement but as the mid-side nodes have been constrained to be related to the corner nodes the element edges have a linear variation of displacement. Thus, in fact, Wilson utilized a constrained linear strain quadrilateral. Similarly the rectangular element used by Khanna has a quadratic variation of displacement within the element but a linear variation along the edges. This element also has the disadvantage that it is awkward to use in systems with irregular boundaries and gradation of mesh within the structure is very difficult to achieve.

There is also a disadvantage in the use of finite elements with only displacement degrees of freedom at the nodes if it is desired to incorporate flexural beam members within the structure and with soil-structure programs this is a very desirable feature. The frame structure has naturally three degrees of freedom at each joint or node, two displacements and a joint rotation. This would mean that some nodes in the combined structure would require extra degrees of freedom and this would tend to lead to complexities in the direct stiffness assemblage of the total stiffness matrix of the system. Provided only a small number of nodes are affected it is easier to accommodate different element types if they only utilize some of the degrees of freedom as the remaining degrees of freedom with no stiffness contributions can be easily constrained to be zero. This would be the case in soil-structure analyses as the frame would be a small part of the total system.

The Quadratic Strain Quadrilateral

The authors decided to utilize a refined plane-strain, plane-stress finite element which has six degrees of freedom at each of the nodal points, they are the two displacements in the x and y directions, u and v respectively, and the derivatives of u and v with respect to both x and y. As a lumped mass formulation was to be used masses would only be associated with the displacements u and v and therefore though each node would have six degrees of freedom for the stiffness formulation it would have only two dynamic degrees of freedom which would have to be accommodated in the eigenvalue problem for the natural modes and frequencies.

In order to take the most advantage of element subdivision the quadrilateral element, see Fig. 1, is formed by the direct stiffness combination of four quadratic strain triangles with the internal node located at the centroid of the quadrilateral.

These triangular elements originally developed for shell anaylsis[11,12] are noted for their ability to accommodate extremely high strain gradients and as they have no mid-side nodes to be constrained they permit each element throughout the discretized soil structure system to have a cubic variation of displacement. Further, this displacement field is compatable with the flexural deformation of beam members. This permits a ready combination of these quadrilateral elements with beam elements in the analysis.

Quadratic Strain Triangle (QST)

Within each triangular element the displacements are expressed as a function of the values of the displacements and their derivatives at each of the three nodes as well as the two displacements of a node located at the centroid of each triangular element. Thus each element has twenty kinematic degrees of freedom.

The nodal displacement vector $\{r\}$ is then arranged as follows.

$$\{\bar{r}\}^T = \langle \bar{u}^T \bar{v}^T \rangle$$

with
$$\{\bar{u}\}^T = \langle u_1, u_{x1}, u_{y1}, u_2, u_{x2}, u_{y2}, u_3, u_{x3}, u_{y3}, u_c \rangle$$

$$\{\bar{v}\}^T = \langle v_1, v_{x1}, v_{y1}, v_2, v_{x2}, v_{y2}, v_3, v_{x3}, v_{y3}, v_c \rangle$$

where the subscripts 1,2,3 and c refer to the corner nodes 1,2 and 3 and c is the centroidal node of the element.

$$u_{xi} = \frac{\partial u_i}{\partial x} \quad ; \quad v_{xi} = \frac{\partial v_i}{\partial x} \quad \text{etc.}$$

The displacements at any point within the triangular element are expressed as

$$u(\eta_1, \eta_2, \eta_3) = \{\phi\}^T \{\bar{u}\}$$

$$v(\eta_1, \eta_2, \eta_3) = \{\phi\}^T \{\bar{v}\}$$

where $\{\phi\}$ is an interpolation vector in "natural" or "triangular" co-ordinates η_1, η_2 and η_3 as shown in Fig 2 ($\{\phi\}$ is given in Appendix I).

The linearized strain-displacement equations are then

$$\{\varepsilon(\eta_1, \eta_2, \eta_3)\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} \end{Bmatrix}$$

$$= \begin{bmatrix} \phi_x^T & \cdot \\ \cdot & \phi_y^T \\ \phi_y^T & \phi_x^T \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} = [B]\{\bar{r}\}$$

where the vectors $\{\phi_x\}$ and $\{\phi_y\}$ are the derivatives of $\{\phi\}$ with respect to x and y respectively with

$$\frac{\partial \phi_i}{\partial x} = \frac{1}{2A} \cdot \frac{\partial \phi_i}{\partial \eta_k} \cdot b_k \; ; \quad \frac{\partial \phi_i}{\partial y} = \frac{1}{2A} \cdot \frac{\partial \phi_i}{\partial \eta_k} \cdot a_k \quad \text{etc.}$$

Assuming a linear elastic material the stresses are related to the strains by the constitutive law

$$\{\sigma(\eta_1, \eta_2, \eta_3)\} = [C]\{\varepsilon(\eta_1, \eta_2, \eta_3)\}$$

where $[C]$ is a 3 x 3 matrix and is a function of the elastic constants of the material, (see Appendix II). The matrix $[C]$ may also be a function of η_1, η_2 and η_3.

The internal strain energy U of the element can now be expressed as

$$U = \tfrac{1}{2}\int_{vol} \{\varepsilon\}^T \{\sigma\} d(vol)$$

$$= \tfrac{1}{2}\{\bar{r}\}^T [\int_{vol} [B^T][C][B] d(vol)] \{\bar{r}\}$$

The external work V done by the nodal forces $\{\bar{P}\}$ is

$$V = \{\bar{r}\}^T \{\bar{P}\}$$

From the principle of minimum potential energy

$$\delta(U - V) = 0$$

$$\therefore \quad [\int_{vol} [B^T][C][B]d(vol)]\{\bar{r}\} = \{\bar{P}\}$$

Therefore, by definition, the stiffness matrix of the element is

$$[\bar{k}] = [\int_{vol} [B^T][C][B]d(vol)]$$

The structure of the nodal displacement vector is now rearranged so that

$$\{r\}^T = \langle r_1^T, r_2^T, r_3^T, r_c^T \rangle$$

where $\{r_i\}^T = \langle u_i, v_i, u_{xi}, v_{yi}, u_{yi}, v_{xi} \rangle \qquad i = 1,2,3$

and $\{r_c\}^T = \langle u_c v_c \rangle$

The stiffness matrix is rearranged to correspond to $\{r\}$ by a simple permutation of rows and columns and as there is no mass associated with the centroidal node the degrees of freedom u_c and v_c are removed by a process of static condensation to give the 18 x 18 stiffness matrix $[k]$ associated with the displacement vector $\{r\}$ where

$$\{r\}^T = \langle r_1^T, r_2^T, r_3^T \rangle$$

The four QST finite elements are now combined using the direct stiffness method[13] with the centroidal degrees of freedom of the quadrilateral element arranged at the bottom of the displacement vector and as there will be no mass associated with the centroidal node of the quadrilateral the centroidal degrees of freedom are eliminated by the same static condensation process that was used for the QST. This then results in the 24 x 24 stiffness matrix of the quadrilateral element.

To maintain uniformity of element size regions in the input finite element mesh are also subdivided into three QST elements with the internal node at the centroid.

The mass of each triangular sub-element is assumed to be distributed so that half of the element mass is distributed to each of the external nodes of the quadrilateral associated with each QST element. This, the authors felt, is a better distribution than allocating a quarter of the quadrilateral mass to each external node. If a consistent mass matrix[14] was to be used, the condensation process of the centroidal nodes of both the QST and the quadrilateral elements could not be achieved as all degrees of freedom would have associated masses and as the inertia forces are not known 'a priori' the internal degrees of freedom cannot be eliminated. This would defeat one of the objectives of using this element.

The total stiffness matrix $[K]$ of the system is assembled from the element stiffness by the direct stiffness method. Any load vectors $\{P\}$ are then able to be solved for the nodal displacements $\{R\}$ using the LDU theorem

i.e. $\quad [K]\{R\} = \{P\}$

let $\quad [K] = [L][D][L^T]$

where $[L]$ is lower triangular and $[D]$ is diagonal

then $\qquad [L][D]\{y\} = \{P\}$

and $\qquad [L]^T\{R\} = \{y\}$

which are both simple substitution operations.

Once the nodal displacements are found the element strains and stresses are easily obtained.

It is the reduction of the stiffness matrix that is the most demanding of computer time and precision. The demands on time are even more acute if the matrix is sufficiently large that after making the utmost benefit of the matrices symmetry and bandedness it cannot be completely stored in the core memory and use has to be made of back-up storage such as magnetic disc or tape. It is for this reason that only elastic analyses are contemplated for dynamic studies using the finite element method as every time the material properties of an element change then the total stiffness would have to be reformed and reduced and the computer time for such an earthquake analysis could run to hours or days on anything but the largest and fastest of computers. The next best approach is that of Seed and Idriss[5] who take the initial elastic solution, find the maximum stresses, take a fraction of these, ($\frac{1}{2}$ or $\frac{3}{4}$ etc.) adjust the material constants to these stress levels and repeat the analysis. This process is repeated two or three times and appears to give good results.

Dynamic Analysis

The equations of motion for the discretized system can be written as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = -[M]\{a\}\ddot{u}_g(t)$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices respectively, $\{\ddot{x}\}$, $\{\dot{x}\}$ and $\{x\}$ are the accelerations, velocities and displacements relative to the bedrock, $\{a\}$ is a vector of unit values and $\ddot{u}_g(t)$ is the input acceleration of the bedrock.

For an elastic system, there are two alternative ways of obtaining the response to the accelerogram $\ddot{u}_g(t)$. The first approach is to use the linear acceleration method of Newmark[15,16] where an effective stiffness matrix is formed and a load vector is solved, at every time step, for the instantaneous accelerations, velocities and displacements. Though this approach requires the formulation of only one matrix and its subsequent reduction by the LDU theorem there are a large number of solutions to be obtained for all the time steps in the integration. Newmark showed that the largest time step for the linear acceleration method could be $\frac{1}{4}$ of the shortest period of free vibration within the structure and for a finite element idealization some of the higher modes may have very short periods. Even using the modified linear acceleration method of Wilson[9] the shortest time step that would accurately represent the accelerogram would be of the order of 1/80 second and hence for ten seconds of the input record approximately 800 load vectors would have to be formed and solved. Experience has shown that other integration methods such as Nordsieck[17] are too slow to be of any practical use in these analyses.

The alternative to the direct integration approach is the superposition of the modal responses. In this method the first step is to obtain the natural frequencies and mode shapes of the structure. The equation of motion of undamped, free vibration (simple harmonic motion) leads to the eigenvalue problem.

$$\left([K] - \omega^2 [M]\right)\{x\} = \{0\}$$

The eigenvalue problem is solved for the natural frequencies and the mode shapes $\{V\}$. If the mode shapes are normalized so that

$$[V]^T [M] [V] = \lceil I \rfloor$$

where $\lceil I \rfloor$ is the identity matrix and the damping matrix is assumed to be of the form where

$$[V]^T [C] [V] = \lceil 2\lambda\omega \rfloor$$

which also satisfies the usual assumption of methods using direct integration(9). The equation of motion now becomes

$$\lceil I \rfloor \{\ddot{b}\} + \lceil 2\lambda\omega \rfloor \{\dot{b}\} + \lceil \omega^2 \rfloor \{b\} = -[V]^T [M] \{a\} \ddot{u}_g(t)$$

where $\{x\} = [V]\{b\}$

This is a set of uncoupled equations of motion for each mode. They can be integrated by any of the methods of numerical integration but the most common method is that of the Linear Acceleration Method of Newmark(15) which is simple and as only the first few modes, up to say the first twenty, are used the period of the highest mode is still fairly large and reasonable time steps may be utilized. Also Newmark's method of integration has the advantage that the results are very stable until the time step is increased beyond the allowable limit and then catastrophic divergence is observed, whereas Wilson's method, which is very stable, slowly diverges from the real solution as the time step increases with no indication of divergence and thus the user may be unaware that the time step being used is too great for accuracy.

Other advantages of this method are :-

(1) The natural periods of the structure are obtained and these can be used for checking against response spectra.

(2) Once the modal properties have been obtained, the response to any number of different earthquake accelerograms can be obtained with the minimum amount of effort.

(3) More is usually known about modal damping than about the actual damping coefficients in the damping matrix, in this method only the damping ratio for each mode need be provided.

(4) Only those modes that are considered significant need be used to obtain the response to the earthquake and hence the computational effort can be considerably reduced.

The major difficulty, however, with this method of analysis is the necessity of finding the mode shapes and frequencies as this involves the solution of a relatively large eigenvalue problem.

There are two approaches to the solution for the natural frequencies and mode shapes. The first is a direct method and does not require any knowledge of the approximate form of the mode shapes but does impose a fairly severe restriction on the number of nodal points within the system being analysed.

As a lumped mass idealization is being used, inertia forces are only associated with the displacement degrees of freedom and the derivative degrees of freedom can be eliminated from the system of equations by a static condensation. However, though this approach is efficient at the element level, it is not a practical approach when dealing with the total stiffness matrix of the structure as it would involve large matrix inversions and the destruction of the banded nature of the stiffness matrix. The condensation can most efficiently be achieved by forming a condensed flexibility matrix by solving a static analysis for a unit load on each displacement degree of freedom in the structure and selecting from the flexibility vector only those degrees of freedom associated with tne displacements. This flexibility matrix is of the order 2N where N is the number of unrestrained nodal points in the system, whereas the order of the original stiffness is 6N where N is the total number of nodal points in the system. In terms of this condensed flexibility matrix the equation of motion of free vibration becomes

$$[F]\lceil M \rfloor \{x\} - \frac{1}{\omega^2}\{x\} = \{0\}$$

Let $\{y\} = \lceil M \rfloor^{\frac{1}{2}} \{x\}$

and multiply by $\lceil M^{\frac{1}{2}} \rfloor$ leads to

$$[F^*]\{y\} - \frac{1}{\omega^2}\{y\} = \{0\}$$

where $[F^*]$ is symmetric.

After solving for ω and $\{y\}$ the mode shapes $\{x\}$ can be obtained from

$$\{x\} = \lceil M^{-\frac{1}{2}} \rfloor \{y\}$$

With a digital computer core memory of 32k words the maximum size of $[F^*]$ that can be accommodated is approximately of order 160 to 170 which means 80 to 85 unconstrained nodal points and probably a total number of nodal points (including those on the boundary) being approximately 100.

For systems requiring more nodal points than this for an adequate idealization or for a more refined mesh for accuracy sensitivity analysis the Modified Rayleigh-Ritz method is used. In this approach a set of approximate mode shapes $[G]$ is selected. These may be arrived at by experience, from previous analyses, or by use of the direct method of finding mode shapes on a coarse mesh finite element idealization. As the Modified Rayleigh-Ritz method improves this set of trial modes prior to using the Rayleigh-Ritz method the choice of the trial shapes is not as critical as with the Rayleigh-Ritz approach. The deflection is assumed as

$$\{\bar{x}\} = [G]\{\alpha\}$$

then the inertia forces in free vibration are

$$\{p\} = -\omega^2[M]\{x\} = -\omega^2[M][G]\{\alpha\}$$

and the resulting displacements or improved trial mode shapes $\{x\}$ are found from a static analysis

$$[K]\{x\} = -\omega^2[M][G]\{\alpha\}$$

i.e. $$\{x\} = -\omega^2[K^{-1}][M][G]\{\alpha\} = -\omega^2[B]\{\alpha\}$$

The equations of motion of free vibration are now rewritten as

$$[K^{**}]\{\alpha\} - \omega^2[M^{**}]\{\alpha\} = \{0\}$$

where $$[K^{**}] = [B^T][K][B] = [B^T][M][G]$$

and $$[M^{**}] = [B^T][M][B]$$

It will be noticed that $[K^{**}]$ does not require the stiffness matrix $[K]$ as this was destroyed during the LDU reduction required to obtain the improved mode shapes.

If $[H]$ is the normalized mode matrix of $[M^{**}]$ then $\lceil M^* \rfloor$ given by

$$\lceil M^* \rfloor = [H^T][M^{**}][H]$$

is a diagonal matrix with the eigenvalues of $[M^{**}]$ arranged down the diagonal.

From this

$$\lceil M^{*-\frac{1}{2}} \rfloor^T[H^T][M^{**}][H]\lceil M^{*-\frac{1}{2}} \rfloor = \lceil I \rfloor$$

and $$\lceil M^{*-\frac{1}{2}} \rfloor^T[H^T][K^{**}][H]\lceil M^{*-\frac{1}{2}} \rfloor = [K^*]$$

where $\lceil I \rfloor$ is the identity matrix and $[K^*]$ is symmetric

with $$\{\alpha\} = [H]\lceil M^{*-\frac{1}{2}} \rfloor\{f\}$$

The equation of motion now becomes

$$[[K^*] - \omega^2\lceil I \rfloor]\{f\} = \{0\}$$

and can be solved by any of the standard eigenvalue routines for symmetric real matrices. If $[F]$ is the normalized mode matrix of $[K^*]$ then

$$[A] = [H]\lceil M^{*-\frac{1}{2}} \rfloor[F]$$

is the matrix of normal modes since

$$[A^T][M^{**}][A] = \lceil I \rfloor$$

and $$[A^T][K^{**}][A] = \lceil \omega^2 \rfloor$$

and the mode shapes in the original x,y co-ordinates follow as

$$[V] = [B][A]$$

Though this method required two eigenvalue solutions, the number of trial modes is usually quite small and hence the eigenvalue problems are of small order and require little computational effort.

Once the mode shapes have been obtained then the stress patterns associated with the displacement shapes are also readily computed.

The Computer Analysis

The program developed for the soil structure analyses is designed to be able to use either method of solving the eigenvalue problem and then to obtain the response of the system to the input accelerograms. At present the maximum number of nodal points (with a digital computer having a 32k core memory) is 200 though in the use of the direct eigenvalue method of limit of 60 unconstrained nodal points is enforced. However, future development will increase the number of nodal points to approximately 300 and the number of degrees of freedom for the direct eigenvalue approach to approximately 170. Up to 20 mode shapes can be used in the analysis and the response to two components, horizontal and vertical, of the earthquake record can be computed, and then combined to obtain the complete response.

The input data is simplified as much as possible using data interpolation and generation routines and all input is in 'free-format' on punched cards. Output includes contour diagrams of displacements and stresses of the mode shapes and the response as well as the usual tabulated numerical data. The response part of the program can output the total acceleration on the relative accelerations, the relative displacement and the stresses in the system. The program can also execute a static load analysis.

Taramakau River Bridge Site

The site is of a 1,000 foot bridge located approximately in the centre of the 6,000 wide valley floor. Preliminary geophysical surveys indicated a minimum thickness of gravel of approximately 520 feet. Two simplified extreme estimates of the valley cross-section are shown in Figure 3 together with the elastic properties of the gravel. As the section is symmetrical and the loading from the horizontal component of the earthquake is antisymmetric there is no differential horizontal displacement between the ends of the bridge, but there is a differential displacement between the centre and the ends of the bridge of approximately 0.5 inches. The fundamental frequencies of the 520 and 1,000 foot thicknesses are 0.313 and 0.206 cycles/second respectively. From the simple theory for a single shear layer of thickness t, shear modulus G and density ρ in free vibration, the frequency is given by

$$f = 1/(4t).\sqrt{(G/\rho)}$$

and is 0.3125 and 0.165 cycles/second respectively. The first section is wide relative to its depth and the shear beam analogy is fairly realistic but in the second section the greater depth of the valley has a more marked effect on the frequency. Analyses which are to include the bridge itself are to be carried out in the near future.

Table A shows a comparison of the natural frequencies of the first three anti-symmetric modes and also the deflections resulting from 10 seconds of the accelerogram of the North-South component of El-Centro, May 1940 earthquake.

Frame on a Shallow Foundation

The frame shown in Figure 4 is that

analysed by Khanna[10]. However, as beam members have not been incorporated in the program as yet, the frame has been idealized as an assemblage of finite elements, the thicknesses and depths being adjusted to give the correct sectional properties. The first two mode shapes are shown in Figure 5 and the natural frequencies are compared with those obtained for a rigid foundation. Khanna, who had pinned supports for the frame at ground level, a consequence of the simple elements used, found the first two natural frequencies to be 1.22 and 4.17 cps respectively for the frame on the soil layer and 1.24 and 5.25 respectively for the rigid foundation.

The authors used a built-in support at ground level and hence the natural frequencies found by Khanna, for the rigid soil case, are, as expected, a little lower. Also, as the finite elements used in the analysis are much more flexible than those used by Khanna, together with a more refined mesh, the soil has a slightly more pronounced effect on the frequencies than that found in the earlier analyses.

References

(1) Shepherd, R., Donald, R.A.H. "Foundation Deformation Effects in Structural Analysis". Proc. 5th Australia-New Zealand Conf. on Soil Mechanics and Foundation Engineering, Auckland, Feb. 1967.

(2) Palmer, D. J., "The Effect of Ground Compliance on the Free Vibration of Shear Buildings". N.Z. Engineering, Nov. 1969.

(3) Idriss, I. M., Seed, H. B., "Seismic Response of Soil Deposits". Journal Soil Mechanics Div. ASCE. Vol. 96 No. SM2. 1970.

(4) Seed, H. B., Idriss, I. M., "Analysis of Ground Motions at Union Bay Seattle During Earthquakes and Distant Nuclear Blasts". Bulletin Seismological Soc. of America, Vol. 60, No. 1, Feb. 1970.

(5) Seed, H. B., Idriss, I. M., "Influence of Soil Conditions on Ground Motions During Earthquakes". J. Soil Mechanics and Foundation Engineering Div., ASCE. Vol. 95, No. SM1 Jan. 1969.

(6) Clough, R. W., "The Finite Element Method in Plane Stress". Proc. 2nd ASCE Conf. on Electronic Computation, Pittsburgh, Pennsylvania, Sept. 1960.

(7) Wilson, E. L., "Structural Analysis of Axisymmetric Solids". AIAA Journal, Vol. 3, No. 12 Dec. 1965.

(8) Clough, R. W., "Comparison of Three Dimensional Finite Elements". Proc. ASCE Symposium on Application of Finite Element Methods in Civil Engineering. Nashville, Tennessee, Nov. 1969.

(9) Wilson, E. L., "A Method of Analysis for the Evaluation of Foundation Structure Interaction". Proc. 4th World Conf. on Earthquake Engineering, Santiago, Chile, Jan. 1969.

(10) Khanna, J., "Elastic Soil Structure Interaction". Proc. 4th World Conf. on Earthquake Engineering, Santiago, Chile, Jan. 1969.

(11) Carr, A. J., "A Refined Finite Element Analysis of Thin Shells Including Dynamic Loadings". Ph.D. Dissertation, University of California, Berkeley, May 1967.

(12) Carr, A. J., Clough, R. W., "Dynamic Earthquake Behaviour of Shell Roofs". Proc. 4th World Conf. Earthquake Engineering, Santiago, Chile, Jan. 1969.

(13) Turner, M. J., "The Direct Stiffness Method of Structural Analysis". AGARD Meeting, Aachen, Germany 1959.

(14) Archer, J. S., "Consistent Mass Matrix for Distributed Mass Systems". J. Structural Div. ASCE, Vol. 89, No. ST4, August 1963.

(15) Newmark, N. M. "A Method of Computation for Structural Dynamics". J. Engineering Mechanics Div. ASCE, Vol. 85, No. EM3, July 1959.

(16) Wilson, E. L., Clough, R. W., "Dynamic Response by Step by Step Matrix Analysis". Symposium on the Use of Computers in Civil Engineering, Lisbon, 1962.

(17) Nordsieck, A., "On Numerical Integration of Ordinary Differential Equations". Maths Comput. 16, pp 22-49, 1962.

Appendix I

The displacement interpolation vector $\{\phi\}$ for the Quadratic Strain Triangle finite element (QST).

$$\{\phi\} = \begin{Bmatrix} \eta_1^2(\eta_1+3\eta_2+3\eta_3)-7\eta_1\eta_2\eta_3 \\ \eta_1^2(a_3\eta_2-a_2\eta_3)+(a_2-a_3)\eta_1\eta_2\eta_3 \\ \eta_1^2(b_2\eta_3-b_3\eta_2)+(b_3-b_2)\eta_1\eta_2\eta_3 \\ \eta_2^2(\eta_2+3\eta_3+3\eta_1)-7\eta_1\eta_2\eta_3 \\ \eta_2^2(a_1\eta_3-a_3\eta_1)+(a_3-a_1)\eta_1\eta_2\eta_3 \\ \eta_2^2(b_3\eta_1-b_1\eta_3)+(b_1-b_3)\eta_1\eta_2\eta_3 \\ \eta_3^2(\eta_3+3\eta_1+3\eta_2)-7\eta_1\eta_2\eta_3 \\ \eta_3^2(a_2\eta_1-a_1\eta_2)+(a_1-a_2)\eta_1\eta_2\eta_3 \\ \eta_3^2(b_1\eta_2-b_2\eta_1)+(b_2-b_1)\eta_1\eta_2\eta_3 \\ 27\eta_1\eta_2\eta_3 \end{Bmatrix}$$

Appendix II

Constitutive matrix $[C]$ where E = Young's Modulus and v = Poission's Ratio.

Isotropic Plane Strain:

$$E/((1-2v)(1+v)) \begin{bmatrix} 1-v & v & . \\ v & 1-v & . \\ . & . & (1-2v)/2 \end{bmatrix}$$

Isotropic Plane Stress:

$$E/(1-v^2) \begin{bmatrix} 1 & v & . \\ v & 1 & . \\ . & . & (1-v)/2 \end{bmatrix}$$

List of Symbols

$[\ ]$ Matrix (Superscript T denotes transpose of matrix)

$\lceil\ \rfloor$ Diagonal matrix

$\{\ \}$	Column vector
$\langle\ \rangle$	Row vector
A	Element area, normal mode matrix
B	Strain-Displacement matrix for finite elements
C	Stress-Strain matrix
D	Diagonal matrix
F	Condensed flexibility matrix
G	Matrix of trial mode shapes
H	Eigenvector matrix of generalised mass matrix
I	Identity matrix
K	Stiffness matrix for structure
L	Lower triangular matrix
M	Mass matrix
P	Nodal point forces for structure
R	Nodal point displacements for structure
U	Internal strain energy
V	External work done, matrix of eigenvectors
a	x dimension of element, unit vector
b	y dimension of element, generalized displacement vector
c	Subscript for centroidal node of finite element
f	Eigenvector, frequency of vibration cps
i	Subscript, corner nodes of finite elements
k	Finite element stiffness matrix, subscript
p	Nodal point forces on finite element
r	Nodal point displacements in finite element
t	Time variable, thickness of soil layer
u	Displacement in x direction
v	Displacement in y direction
x	Cartesian co-ordinate, displacement vector
y	Cartesian co-ordinate, dummy vector, eigenvector
$\ddot{u}_g$	Acceleration of ground or bed-rock
vol	Volume of finite element
α	Generalized co-ordinate, participation of each trial mode in total displacement
ε	Strain, normal strains
γ	Shear strain
η	Triangular co-ordinate for finite element
λ	Critical damping ratio
ρ	Density of Soil
ϕ	Interpolation function, interpolation vector
ω	Natural frequency of vibration (radians/sec)

$$\dot{x} = \frac{\partial x}{\partial t} \; ; \qquad \ddot{x} = \frac{\partial^2 x}{\partial t^2}$$

Result of Analysis		Section I	Section II
Natural Frequencies cps. (Lateral modes only)	1st mode	0.3127	0.2064
	3rd mode	0.4243	0.3345
	5th mode	0.5187	0.3906
Max.Horizontal Displ. at A (ft.)		0.295	0.245
Max.Vertical Displ. at B (ft.)		0.032	0.077
Max. Differential Displ. A-B (ins.)		0.396	0.540

Results from 10 seconds of El-Centro May 1940 North-South.

TABLE A

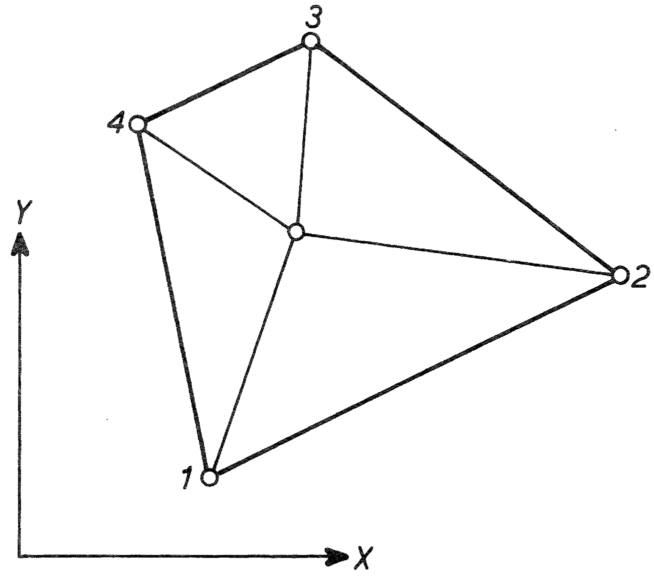
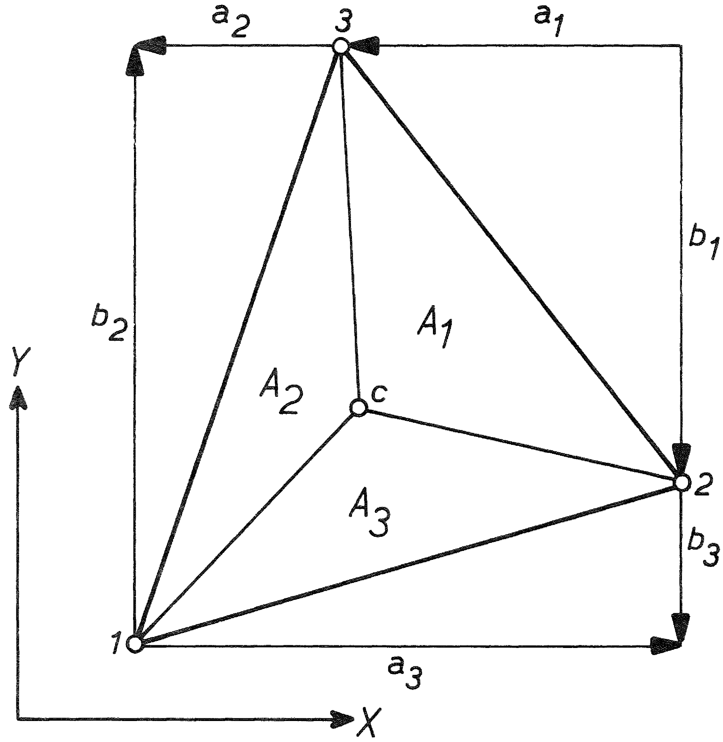
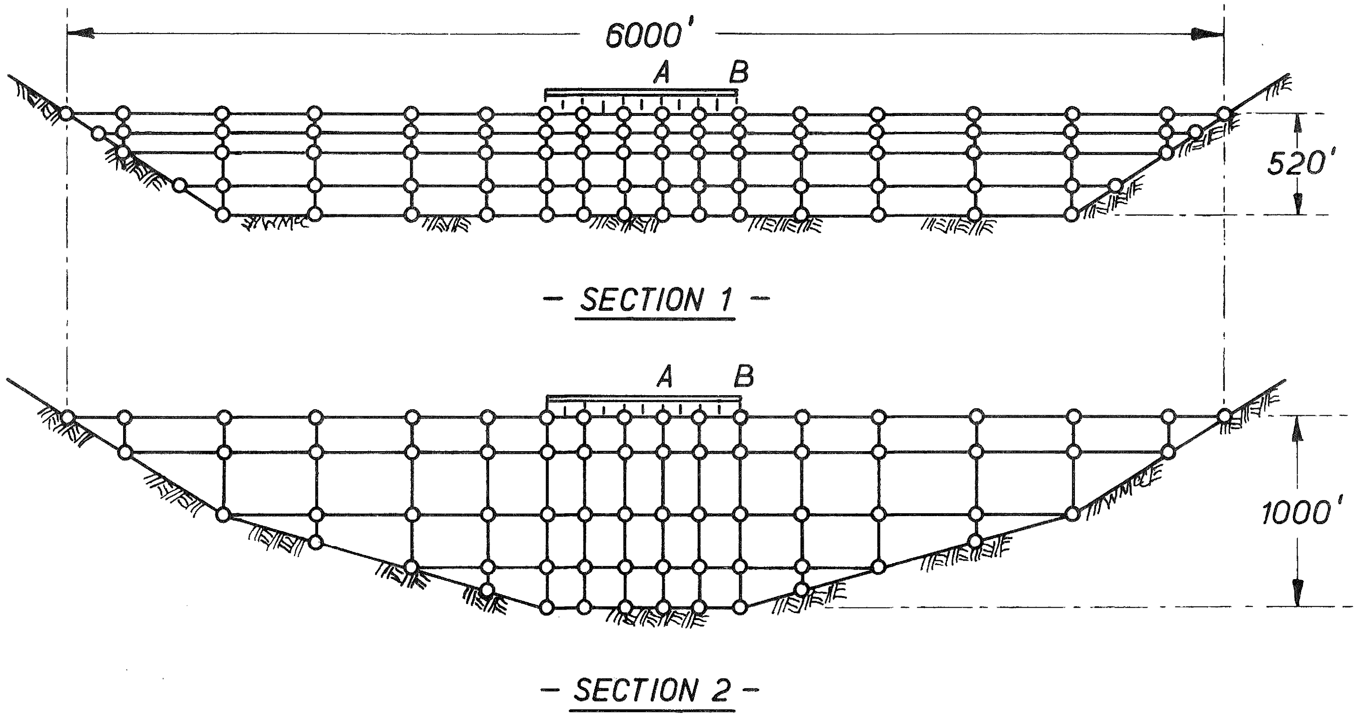


FIGURE 1—Quadrilateral element from 4 triangular elements.



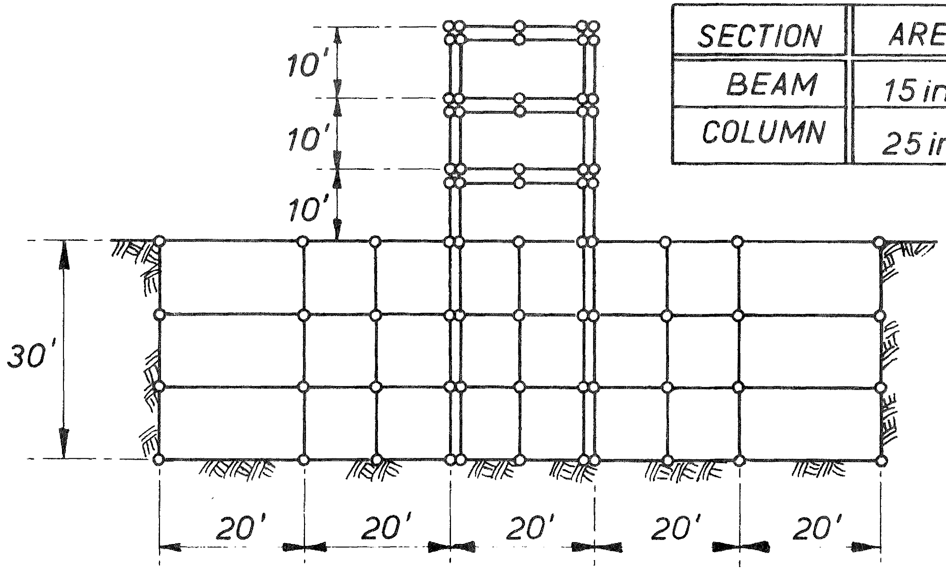
$2A = a_3 b_2 - a_2 b_3$ etc; $\eta_i = A_i/A$

FIGURE 2—QST element



Assumed properties of gravel; E = 4030,000 lb/sq.ft.
G = 1440,000 lb/sq.ft.
V = 0.4 , density = 120 lb cu.ft.

FIGURE 3 —Taramakau River Bridge Site, Westland.

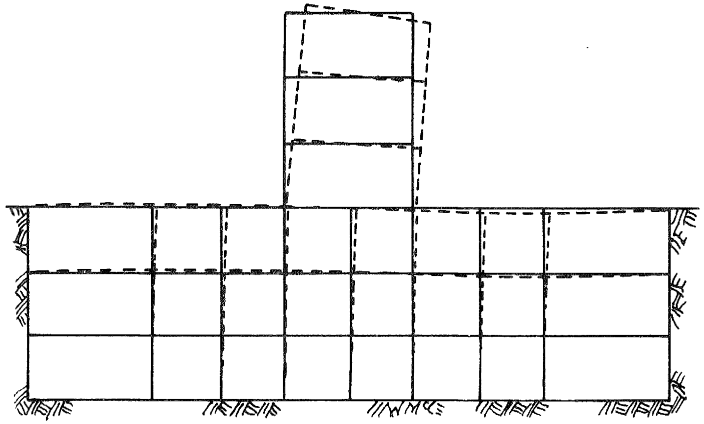


SECTION	AREA	M of I
BEAM	15 in.^{2}	800 in.^{4}
COLUMN	25 in.^{2}	960 in.^{4}

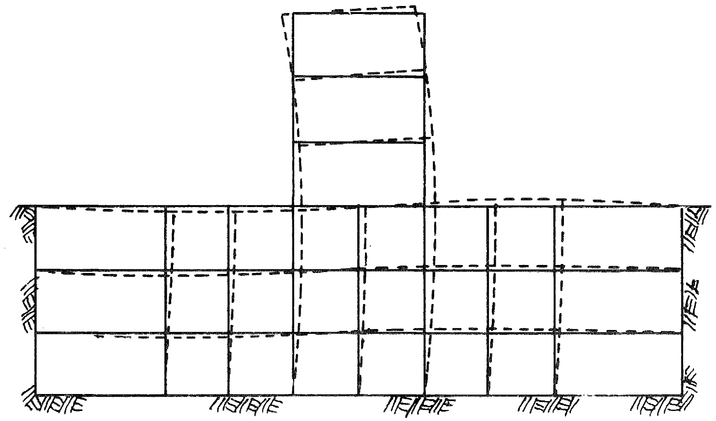
Frame :- E = 30,000 ksi
ν = 0.3
Weight = 30 kips at each beam column joint.

Soil :- E = 10 ksi
ν = 0.4
Density = 130 pcf
Thickness of plane strain elements = 20 ft.

FIGURE 4 — Frame on Shallow Foundation.



1st. mode, Frequency = 1.115 cps
(c.f. Rigid base f = 1.354)



2nd. mode, Frequency = 4.358 cps
(c.f. Rigid base f = 6.548)

FIGURE 5 — First Two Mode Shapes for Three Storey Frame on Shallow Foundation.