NORMAL MODES OF A "CYLINDRICAL VALLEY" OF ALLUVIUM

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ABSTRACT

Some normal modes of vibration are deduced for a cylindrical volume of high bulk modulus, low shear modulus material, embedded in an infinite half space of rigid material. The manner in which they may be excited by travelling waves in the rigid material is examined. The relevance of such processes is discussed with regard to the enhancement of structural damage on soft soil during an earthquake.

THEORETICAL DEVELOPMENT

Consider a cylindrical cavity in an infinite half space of rigid material as shown in Figure 1. The cavity has its axis vertical and is of radius R and depth h. Let the material be infinitely rigid in both a compressional and a shear sense, and let the cavity be filled with a homogeneous, isotropic, linear material of infinitely great bulk modulus but of finite shear modulus μ. That is, it can not be compressed but it can be sheared. Let the flexible material have a density of ρ.

Let there be a standing wave of shear strain set up within the flexible material, specified by a time varying vector field of displacement $\underset{\sim}{u}(t)$. According to Bullen [1], the equation

$$\rho \frac{\partial^2}{\partial t^2}(\nabla \times \underset{\sim}{u}) = \mu \nabla^2 (\nabla \times \underset{\sim}{u}) \qquad (1)$$

must be satisfied over the volume. Because the material is incompressible we will also require

$$\nabla . \underset{\sim}{u} = 0 \qquad (2)$$

over the volume. We will require

$$\underset{\sim}{u} = 0 \qquad (3)$$

at the boundaries as there is no slip.

The natural coordinate system to use in this case is the cylindrical one, with coordinates referred to r, θ and z axes.

At the top surface, (z=h), we require

$$\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} = 0 = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \qquad (4)$$

to ensure no shear strain in any vertical plane at the free surface.

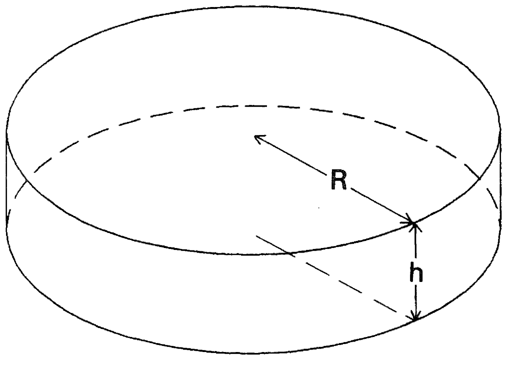

FIG. 1. GEOMETRY OF THE CYLINDRICAL CAVITY.

A set of solutions of equation (1) subject to the incompressibility condition (2) and boundary conditions (3) and (4) is

$$\underset{\sim}{u}(t) = \hat{\underset{\sim}{a}}_\theta \frac{\sin (2n-1)\pi z}{2h} J_1(kr) e^{-i\omega_{s,n} t} \qquad (5)$$

with

$$k = \frac{j_{1,s}}{R} \quad \text{and} \qquad (6)$$

$$\omega_{s,n} = v \sqrt{\frac{(2n-1)^2 \pi^2}{(2h)^2} + \frac{(j_{1,s})^2}{(R)^2}} \qquad (7)$$

Where:

$\hat{\underset{\sim}{a}}_\theta$ is the unit θ vector,

J_1 is a first-order Bessel function,

$j_{1,s}$ is the s-th root of J_1, and may be obtained from tables [2],

$v = \left(\frac{\mu}{\rho} \right)^{\frac{1}{2}}$ is the shear wave velocity in the medium,

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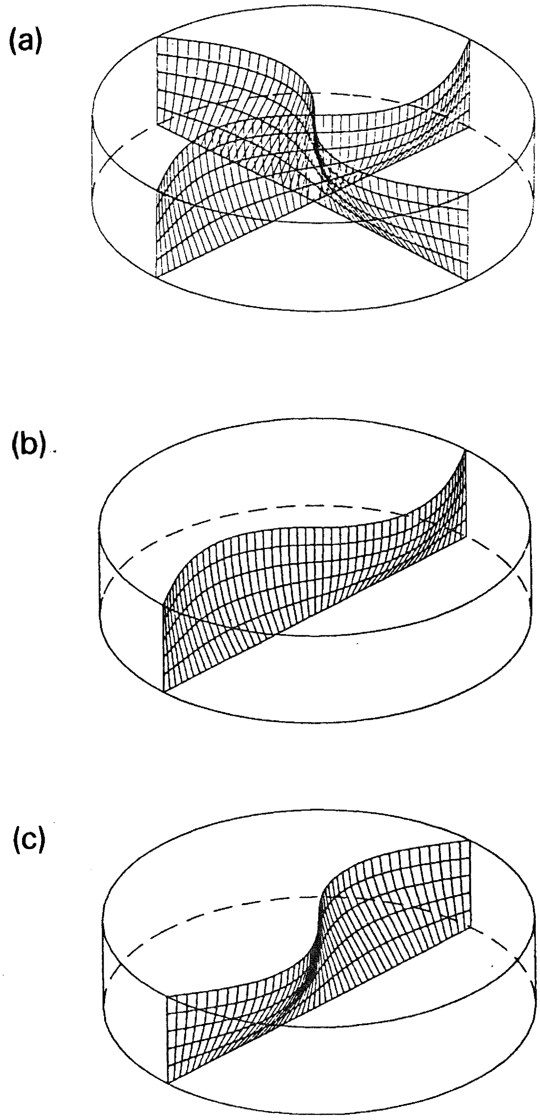

FIG. 2. THE FUNDAMENTAL MODE.
(a) SHOWING CIRCUMFERENTIAL NATURE OF THE DISPLACEMENT.
(b) MAXIMUM DISPLACEMENT.
(c) ONE HALF PERIOD AFTER MAXIMUM DISPLACEMENT.

ω_s is the angular frequency for the s-th mode of vibration,

n and s are integers,

μ is the shear modulus of the infill medium, and

ρ is the density of the infill medium.

These solutions may be verified by using the relations (e.g., [3])

$$\nabla . \underset{\sim}{u} = \frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \qquad (8)$$

$$\nabla x \underset{\sim}{u} = \hat{\underset{\sim}{a}}_r\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right) + \hat{\underset{\sim}{a}}_\theta\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + \hat{\underset{\sim}{a}}_z\left(\frac{1}{r}\frac{\partial r u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) \qquad (9)$$

$$\nabla^2 \underset{\sim}{u} = \nabla(\nabla . \underset{\sim}{u}) - \nabla \times (\nabla \times \underset{\sim}{u}) \qquad (10)$$

where $\underset{\sim}{u}$ is any vector,

and the relations (3)

$$\frac{d\left(J_1(r)\right)}{dr} = J_0(r) - \frac{J_1(r)}{r} \qquad (11)$$

$$\frac{d}{dr}\left(J_0(r)\right) = -J_1(r) \qquad (12)$$

where J_0 is a zero-order Bessel function.

The solutions given in equations (5), (6) and (7) express the existence of a set of normal modes (or standing waves) within the cylinder. These modes have a sine wave dependence vertically and a first-order Bessel function (which somewhat resembles a damped sine wave) dependence radially.

For the solutions, the displacement is always circumferential, is a maximum at the free surface, and zero at the z axis. All displacement is oscillatory with time, and circumferential, and should be thought of as oscillating along an arc of a circle at any point. A displaced surface for the fundamental mode (n=1, s=1) is portrayed in Figure 2. It has an angular natural frequency of oscillation ω, with

$$\omega = v\sqrt{\left(\frac{\pi}{2h}\right)^2 + \left(\frac{3.83171}{R}\right)^2}$$

In the case of a shallow cylindrical cavity (R>>h) it is to be noted that the values of ω cluster in families just above

$$\omega = v.\frac{2n-1}{2h}\pi$$

for each n, which are the values for an infinite layer of thickness h resonating with plane shear waves. Thus the first family (n=1) has values just above

$$\omega = \frac{v\pi}{2h}$$

and these values of ω increase as s increases, i.e., as the radial pattern has more nodes. The family members will all have a basic quarter sine wave dependence with z.

The foregoing analysis, which closely parallels other solutions of the wave equation in cylindrical coordinates, establishes the existence of normal modes of vibration in a cylindrical cavity of flexible material embedded in rigid material. The solutions may be equally thought of either as standing waves or as a set of eigenfunctions and eigenvalues.

Establishing the existence of normal modes is a necessary step in discussing resonance effects but by itself is not sufficient. For resonance effects to be observed, there must by a mechanism by which the normal modes can be excited.

A common view (Lomnitz and Rosenblueth [4], section 6.5.3) of the response of soils to earthquakes is that they are excited by vertically travelling shear waves. This is often an adequate approach, but is misleading in the context of modes possessing horizontal variations in their mode shapes. The physical situation is that the incoming waves are refracted so that they propagate nearly vertically (as a consequence of their velocities usually being lower near the surface). However the propagation direction is tilted from the vertical in a direction away from the epicentre of the earthquake, by an amount that results in the surface disturbance moving away from the epicentre at the velocity of the waves in deep rock. That is, the surface disturbance has a phase velocity equal to the velocity at depth. Seismologists are able to locate earthquakes as a result of waves apparently travelling horizontally along the surface of the earth (p-waves at around 8 km/s and s-waves at around 4 km/s even though both p and s arrive travelling nearly vertically.

In the analysis that follows, the passage of this disturbance across the surface will be shown to be well suited to excite the modes described earlier.

Suppose that the rigid infinite half space is now not infinitely rigid, but has a shear wave velocity v_r, and that the material filling the cavity has a shear wave velocity v_s. Let a horizontally polarised plane shear sine wave travel horizontally across the half space, and let it have an angular frequency equal to the fundamental mode frequency of the cavity, $\omega_{1,1}$.

If the dynamic displacements in the rigid material are very much smaller than the dimensions of the cylinder, its shape will effectively remain cylindrical and the normal mode analysis performed previously will apply. Essentially the cavity will function in the same way as when no wave is present because its deviations from the static shape will be small. The wavelength in the more rigid material will be

$$\frac{2\pi v_r}{v_s\sqrt{\left(\frac{\pi}{2h}\right)^2 + \left(\frac{3.83171}{R}\right)^2}}$$

There now emerges the possibility of the plane shear wave having a net driving influence on the resonant mode. Using d'Alembert's principle, the sine wave of displacement may be interpreted as a sine wave of acceleration and then as an equivalent sine wave of force, both acting on the masses of material contained in the cylinder. It is readily seen from Figure 3 that the lag as the wave passes through the stiff material enables this equivalent force to drive the fundamental mode, because the wave shape in the stiffer material somewhat resembles the shape of the mode. On alternate half cycles there will be a net clockwise, then anticlockwise, force driving the mode, as the wave moves through one cycle in the stiff material. For a shallow cylinder

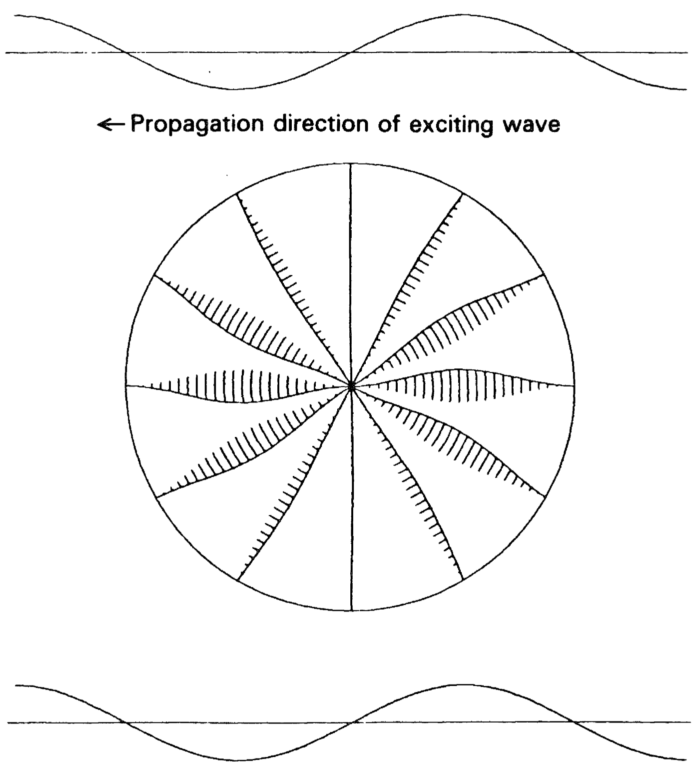

FIG. 3. EXCITATION OF THE FUNDAMENTAL MODE BY A TRAVELLING WAVE.

(R>>h) the wavelength in the stiff material is equal to the diameter of the cylinder when

$$\frac{v_s}{v_r} = \frac{2h}{R},$$

and this would ensure the optimum excitation condition shown in Figure 3.

Thus if a cylinder of depth 20 m and radius 800 m is considered, and shear wave velocities of 4000 m/s in the stiff material and 160 m/s in the flexible material adopted, the situation of Figure 3 will pertain, and the incident travelling wave will excite the fundamental mode in the cylinder, causing a resonant motion resembling an oscillating fluid flow.

If the cylinder diameter were smaller, there would be a weaker excitation of the fundamental mode. If the cylinder diameter were doubled, the second member of the family of resonances, with n=1 and s=2, would be excited as in Figure 4. The chief determinant of the resonant frequency will still be the depth h of the cylinder, and the z-wise variation of the resonant motion will still be a quarter sine wave.

The above has shown that certain modes of vibration may exist in cylindrical cavities filled with material of high bulk modulus and low shear modulus, and that the modes may be excited by plane shear waves propagating horizontally in the surrounding material. The essential features of waves which excite the resonant modes are a phase

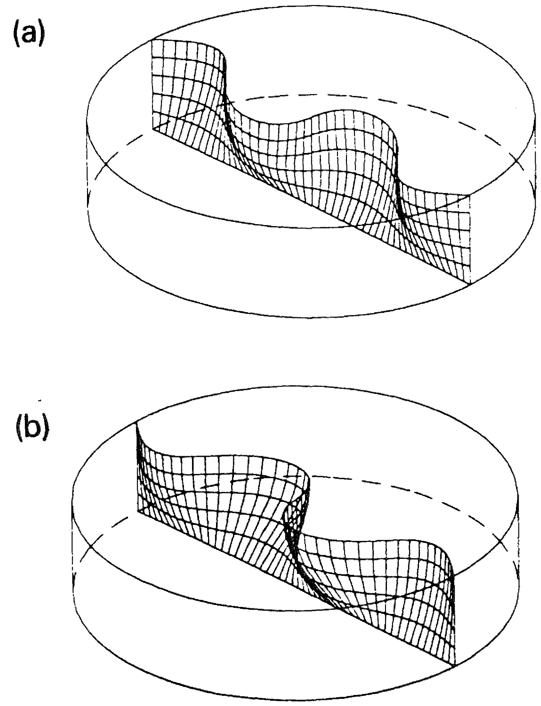

FIG. 4. SECOND MODE.
(a) MAXIMUM DISPLACEMENT.
(b) ONE HALF PERIOD AFTER MAXIMUM DISPLACEMENT.

velocity across the surface, particle motion horizontal and transverse to the propagation direction across the surface, and a quarter wavelength that approximates the depth of the cylinder. This rules out p-waves and Rayleigh waves but allows the possibility of excitation by s-waves and Love waves.

At first sight the system analysed appears to have no relationship to any practical situation. It deals with a simple geometry and with ideal materials in order to obtain an analytical solution. A real situation of interest is the response of a shallow basin of soft soil, perhaps modelled as a cap of a sphere containing a material whose stiffness increases with depth and which has a nonlinear stress-strain curve, and for which the bulk modulus is not infinite, but is high. Jiang, Tong and Kuribayashi [5] have analysed such axisymmetric systems including the case of obliquely arriving waves (ensuring a component of horizontal propagation at the surface) but have not shown modes of the sort described here. At no stage did they consider s_h waves obliquely incident on a wide shallow basin of very flexible incompressible material. Had they done so, modes of the sort described here would be expected.

A problem of nomenclature arises when the issue of bulk modulus is addressed because soils such as those underlying parts of Mexico City are described as "compressible" in order to express the large subsidences which occur as a result of consolidation. However, in terms of undrained dynamic phenomena, such soils have high bulk moduli (reflected in high p-wave velocities) and hence are well modelled as shearable and incompressible in terms of the modes described in this paper.

Real flexible soils depart considerably from the idealisation of a homogeneous linear elastic substance. The particle size typically varies because of the history of deposition, increasing confinement pressure (due to increased overburden) causes the shear modulus to increase (Das [6]) and time under confinement pressure can cause consolidation, reducing void ratio and increasing shear modulus. In addition Das [6] shows that soils are non-linear, generally becoming more flexible under load.

Thus the major differences between the system analysed here and a realistic model of a soil deposit are a shallow cylinder compared with a cap of a sphere, constant shear wave velocity compared with velocity increasing downwards, and linear compared with non-linear stress-strain characteristics. The change from cylindrical to spherical boundaries will cause mode shapes and frequencies to change slightly, as will a velocity increasing with depth. The typical non-linear character of a soil will cause the natural frequencies to decrease as the amplitude of excitation increases. The overall phenomenon of normal modes being excited by horizontally propagating s_h or Love waves, resulting as it does from the high ratio of compressive to bulk modulus, high ratio of width to depth of the deposit, and high ratio of s-wave velocity in rock compared with soil, is expected to occur and give rise to much the same mode shapes. The appropriate rock velocity to use will be the phase velocity across the surface. For waves arriving at near vertical incidence this will be the velocity in deep rock corresponding to the horizontal phase velocity at the surface.

Excitation of modes similar to those considered here, caused in a local deposit of deep flexible soil by a large, distant earthquake, could give rise to the selective failure of buildings within a specific range of heights. Buildings which had the same resonant frequency as the soil and were located near a mode maximum would be subject to increased loadings, whereas those which were non-resonant or located near a node point would be subject to the normally expected loadings for that earthquake.

Such effects are expected for large, distant earthquakes because only a large, distant earthquake radiates energy in the form of many cycles of displacement at low frequencies. In the case of a small, close earthquake, high frequencies are present, and dominate the form of the response spectrum. In the case of a large, close earthquake, damage is universal, and no differences with structural period are noted. In the case of a small, distant earthquake amplitudes are too small to cause damage even when resonantly amplified. But because of the relationship of magnitude with corner frequency, and dispersion and frequency dependent attenuation as the waves propagate, a large, distant earthquake can supply a long-duration train of low frequency waves, ideal for exciting long-period, lightly damped resonant buildings.

Examples of damage contrasts due to large, distant earthquakes acting on deposits of flexible soil are numerous and well known, the most recent being in Mexico City, 1985 [7], [8].

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