SEISMIC RESPONSE OF ELEVATED WATER RESERVOIRS

by B. Hunt* and M.J.N. Priestley*

ABSTRACT:

Equations previously developed for a ground-supported, rigid cylindrical reservoir and based on the classical approximations of linearized, inviscid wave mechanics are extended to cope with the case of elevated reservoirs supported by flexible columns. Solutions for column shear forces and free surface displacements are developed in a form suitable for computer prediction of time-history response. Sensitivity of results to the time-step size and the number of slosh modes is examined, and it is concluded that accurate estimates of water level fluctuations can only be obtained if at least 10 modes are used. A realistic example is presented to illustrate the method, and the response to two accelerograms is compared with predictions obtained from Housner's simplified approach.

INTRODUCTION:

Considerable research effort has been recently directed towards prediction of the response of ground supported water reservoirs or oil storage tanks. The need for such research is apparent from the extent of damage sustained by both concrete and steel reservoirs and tanks in recent earthquakes (8,9,10).

Past research can conveniently be divided into two groups: those dealing with effectively rigid walls, such as is approximated by prestressed concrete cylindrical reservoirs, for example, (4,6,7) and those dealing with tanks where the wall flexibility is high, leading to interaction between tank deformation and fluid pressures. This latter problem, which is characteristic of steel-shell oil storage tanks, has been recently summarised by Clough (2).

Most of the recent research for both categories of problems has been based on earlier theoretical approximations by Jacobsen (7) and, particularly, Housner (4,5). Jacobsen treated the fluid motion as irrotational (potential flow). More recently, comparisons between theory and experiment by Hunt and Priestley (6) and Aslam, Godden and Scalise (1) have shown that an irrotational flow model gives a very good description of scaled model experiments, and, since the relative thicknesses of boundary layers decrease as the scale of an experiment increases, one would expect an irrotational model to give an even better description of the physics for a prototype. Unfortunately, though, Jacobsen neglected free surface displacements and applied an incorrect boundary condition on the free surface. In particular, neglecting free surface displacements leads to the requirement that the normal derivative of the velocity potential vanish on the free surface, but Jacobsen required the potential to vanish on the free surface instead.

Housner (4,5) avoided the problem of integrating the partial differential equation of irrotational flow (the Laplace

equation) by assuming that the incompressible fluid motion could be analyzed as the sum of an "impulsive" motion and a "convective" motion. For the "impulsive" motion, Housner assumed that the free surface remained horizontal, that fluid velocity and acceleration components normal to a vertical plane of motion were zero and that the remaining horizontal velocity component was independent of the vertical coordinate. The unique, irrotational flow solution to this problem states that the entire reservoir of fluid oscillates horizontally as a rigid body. However, Housner's solution gave a horizontal velocity that was a function of the horizontal coordinate, which means that Housner's solution was not for an irrotational motion. In the second part of his analysis for the "convective" motion, Housner assumed that vertical velocity components were identical with the vertical velocity components of a rigid, horizontal plate which was allowed to rotate about a hinge at its geometrical centre. Then use of Hamilton's principle allowed Housner to obtain an estimate for the natural oscillation frequency of these horizontal "plates" of fluid. Finally, Housner stated that the forces calculated from his model were identical with those calculated by placing a particular point mass between horizontal springs at a certain elevation within the empty tank and that higher mode solutions could be found by placing additional masses between horizontal springs within the tank.

Hunt and Priestley (6) calculated the irrotational flow solution that results from using the linearized, free surface boundary conditions of classical wave mechanics. The model required considerable fewer assumptions about the fluid motion than those required for the Housner model, and time-history comparisons between calculated and measured free-surface displacements under simulated seismic motions suggested that the irrotational flow solution modeled the experimental physics very closely. A similar study by Aslam, Godden and Scalise (1) of sloshing in an annular tank gave an equally good comparison between an irrotational flow solution and experimental results, and, since similar comparisons between theory and experiment were not

* Readers, Department of Civil Engineering, University of Canterbury, Christchurch, New Zealand.

made by Housner, it can only be concluded that the irrotational flow solutions can be used with considerably more confidence than the Housner model of springs and point masses.

Since many water reservoirs are placed on flexible towers, a logical extension of this earlier work is to show how the irrotational flow solution obtained by Hunt and Priestley (6) can be coupled with equations that describe the elastic response of a tower to obtain estimates for the support system shear force and free surface displacements. Figure 1 illustrates two different sub-groups of structural type within the classification of elevated water reservoirs. In Fig. 1a, the reservoir is supported by a single central column. Under horizontal base acceleration, the column flexibility will result in both translation and rotation of the reservoir, and a rigorous analysis would need to include the influence of both effects on the response of the reservoir, though it is unlikely that the rotational component of response would be significant.

In the second case, Fig. 1b, with the reservoir supported by a number of columns, rotational response can be ignored and the reservoir analysed for translational response alone. It should be realised that in both cases the problem involves an interaction between support-system stiffness and fluid response that cannot obviously be decoupled: the fluid responds to the displacements of the reservoir rather than of the ground motion, but the reservoir displacements are a function of both ground motion and pressures exerted on the walls by the complex response of the fluid.

PROBLEM FORMULATION:

The problem will be formulated for the class of elevated reservoir illustrated in Fig. 1b, that is, it will be assumed that the base of the reservoir (top of supporting column system) translates under response to seismic attack, but does not rotate. However, as mentioned above, it is not expected that rotation of the column top for the single-column reservoir of Fig. 1a will significantly affect response, so it should be reasonable to apply the method to both classes of structure.

The following additional assumptions are made in the analysis:

- tank walls and base slab are effectively rigid. That is, there is no significant interaction between tank distortion and fluid pressures.
- the column support system remains elastic throughout seismic attack.
- the fluid is inviscid, and boundary-layer effects at the fluid/tank interface are negligible. Thus free-surface fluctuations are undamped.

Although the method will be formulated for the more common case of cylindrical elevated reservoirs, application to

rectangular elevated reservoirs is obvious, making the appropriate adjustments based on the derivations for ground supported rectangular reservoirs in the earlier paper (6).

If the tank rotation is ignored, the problem can be simulated as shown in Fig. 2 by a cylindrical tank rolling on frictionless bearings and connected to the ground by a spring of stiffness, K , representing the columns of the support system. A dashpot, with viscous damping coefficient, C , represents the structural damping.

Calculation of the appropriate value of spring stiffness is a simple exercise in structural mechanics, and is found from the level of horizontal force applied at the level of the tank base for Fig. 1b to induce unit displacement. Thus, for the simple system of n vertical columns without cross-bracing, as illustrated,

$$K = n \cdot \frac{12EI}{h^3} \quad (1)$$

where E is the modulus of elasticity of the column material, I is the moment of inertia of a single column, and h is the column height.

The most significant parameters describing the response will be the total shear force developed in the reservoir support system and the free surface level fluctuation. The formulation is developed to predict these variables.

An application of Newton's second law to the water reservoir shown in Fig. 2 gives

$$F_w - C \dot{x} + K(X - x) = M_r \ddot{x} \quad (2)$$

in which F_w = horizontal water force exerted upon the tank, M_r = reservoir mass, x = horizontal reservoir displacement and X = horizontal displacement of the left spring end = horizontal ground displacement. The dots above an x denote time differentiation, and the water force, F_w , is given by Hunt and Priestley (6) in the following form:

$$F_w = -M_w \left[\ddot{x} + \sum_{n=1}^{\infty} \frac{2 \tanh(\lambda_n D/R)}{(1 - \lambda_n^2) (\lambda_n D/R)} \ddot{x}(\tau) \cos \beta_n x \right] \quad (3)$$

$(t - \tau) \sqrt{\frac{g}{R}} d\tau$

The additional symbols in Eq. 3 are M_w = water mass, D = undisturbed water depth, R = tank radius, g = gravitational constant, t = time, τ = dummy integration variable and

$$\beta_n = \sqrt{\lambda_n \tanh(\lambda_n D/R)} \quad (4)$$

Fig.1. ELEVATED WATER RESERVOIRS

Fig.2. PROBLEM SIMULATION

in which λ_n = nth root of the first derivative of the first-order Bessel function of the first kind $[J'_1(\lambda_m) = 0]$ for $m = 1, 2, 3, \dots$. Since the ground displacement, X , is a known function that can be taken from seismic records, Eq. 2 has $x(t)$ as its only unknown. Equation 2 will be solved by subjecting its solution to the initial conditions

$$\dot{x}(0) = \dot{x}(0) = 0 \quad (5)$$

Equation 5 requires that the tank motion start from a state of rest with a zero initial displacement.

PROBLEM SOLUTION:

A great deal of algebraic clutter can be avoided by introducing the following dimensionless variables:

$$(t^*, \tau^*, \beta_n^*, C^*) = \left[t \sqrt{\frac{K}{M_w + M_r}}, \tau \sqrt{\frac{K}{M_w + M_r}}, \beta_n \sqrt{\frac{g(M_w + M_r)}{KR}}, \frac{C}{\sqrt{K(M_w + M_r)}} \right] \quad (6)$$

Using Eq. 6 in Eqs. 2-3 and eliminating the water force, F_w , gives

$$\ddot{x} + x = X - C\dot{x} - \sum_{n=1}^{\infty} A_n \int_0^t \dot{x}^*(\tau) \cos \beta_n(t - \tau) d\tau \quad (7)$$

in which the asterisk superscript has been omitted for notational convenience and the dimensionless coefficient, A_n , is given by

$$A_n = \frac{2 \tanh(\lambda_n D/R)}{(1 + M_r/M_w)(1 - \lambda_n^2)(\lambda_n D/R)} \quad (8)$$

The variables X and x in Eq. 7 can be made dimensionless by dividing both sides of Eq. 7 by any dimensional constant with units of length. However, this will not be done since it creates no further simplification in Eq. 7.

The solution of the ordinary differential equation

$$\ddot{x} + x = F \quad (9)$$

subjected to the initial conditions given by Eq. 5 is

$$x = \int_0^t F(\tau) \sin(t - \tau) d\tau \quad (10)$$

The substitution of the right side of Eq. 7 for the function $F(\tau)$ in Eq. 10 gives

$$x(t) = \int_0^t \left[X(\tau) - C\dot{x}(\tau) - \sum_{n=1}^{\infty} A_n \int_0^t \dot{x}^*(\xi) \cos \beta_n(\tau - \xi) d\xi \right] \sin(t - \tau) d\tau \quad (11)$$

Equation 11 can be converted to a Volterra integral equation of the second kind by integrating the second term on the right hand side once by parts and by interchanging the order of integration in the third term, computing the inside integral and integrating three times by parts. The result is

$$\alpha x(t) + \int_0^t x(\tau) G(t - \tau) d\tau = \int_0^t X(\tau) \sin(t - \tau) d\tau \quad (12)$$

in which α and G are given by

$$\alpha = 1 + \sum_{n=1}^{\infty} A_n \quad (13)$$

$$G(t - \tau) = C \cos(t - \tau) + \sum_{n=1}^{\infty} \frac{A_n}{1 - \beta_n^2} \left[\beta_n^3 \sin \beta_n(t - \tau) - \sin(t - \tau) \right] \quad (14)$$

Since both the kernel function, $G(t - \tau)$, and the right side of Eq. 12 are continuous, Eq. 12 has the theoretical significance of showing that a unique solution for $x(t)$ exists (Hildebrand (3)). In practical terms this means that a numerical solution of Eq. 12 is likely to be stable and convergent.

Equation 12 has a form that, in theory, allows it to be solved exactly by taking the Laplace transform of both sides. However, the problem of inverting the transform in this case is too difficult to make such an approach realistic. Thus Eq. 12 will be solved numerically.

A numerical solution for $x(t)$ can be carried out by obtaining an equation in which $\dot{x}$ and $\ddot{x}$, rather than x and X , appear within the integrals. Thus, the first term on the right side of Eq. 11 is integrated by parts once, the second term is left unchanged, and the third term is modified by interchanging the order of integration, computing the inside integral and integrating the result twice by parts to obtain

$$x(t) - \int_0^t \dot{x}(\tau) \dot{H}(t - \tau) d\tau = X(t) - \int_0^t \dot{X}(\tau) \cos(t - \tau) d\tau \quad (15)$$

in which

$$H(t - \tau) = C \cos(t - \tau) + \sum_{n=1}^{\infty} \frac{A_n}{1 - \beta_n^2} \left[\beta_n \sin \beta_n(t - \tau) - \sin(t - \tau) \right] \quad (16)$$

Finally, the integration interval from 0 to t can be divided into P equal time steps, x and X can be taken as linear functions within each time step and the integrals can be computed exactly to give

$$x_{p+1} + \sum_{i=1}^P \frac{x_{i+1} - x_i}{\Delta t} \left[H(t_{p+1} - t_{i+1}) - H(t_{p+1} - t_i) \right] = x_{p+1} + \sum_{i=1}^P \frac{x_{i+1} - x_i}{\Delta t} \left[\sin(t_{p+1} - t_{i+1}) - \sin(t_{p+1} - t_i) \right] \quad (17)$$

Since $x_1 = X_1 = 0$, Eq. 17 can be rearranged to obtain a recurrence formula for x :

$$x_{p+1} = \left[UX_{p+1} + \sum_{i=2}^P (X_i S_i - x_i T_i) \right] / V \quad (18)$$

The coefficients S_i , T_i , U and V are given by

$$S_i = \left[2 \sin(t_{p+1} - t_i) - \sin(t_{p+1} - t_{i-1}) - \sin(t_{p+1} - t_{i+1}) \right] / \Delta t \quad (19)$$

$$T_i = \left[2H(t_{p+1} - t_i) - H(t_{p+1} - t_{i-1}) - H(t_{p+1} - t_{i+1}) \right] / \Delta t \quad (20)$$

$$U = 1 - \frac{\sin(\Delta t)}{\Delta t} \quad (21)$$

$$V = 1 + \frac{C - H(\Delta t)}{\Delta t} \quad (22)$$

in which Δt = constant time step and

$H(\Delta t) \equiv H(t_{p+1} - t_p)$ is given by Eq. 16.

Equation 18 permits the time-history of the horizontal reservoir displacement to be calculated from the time-history of the ground displacement, and the difference of these two displacements can be multiplied by the spring stiffness, K , to obtain the support system shear force. Free surface displacements can also be obtained by calculating numerical values for the second derivative of the reservoir displacement for use in Eqs. 29-30 of the earlier paper by Hunt and Priestley (6). It is worth pointing out, however, that both of these calculations require a double summation, and the cost of the calculation increases rapidly as the time step is decreased and the number of modes is increased.

APPLICATION:

The method developed in the previous section requires choice of the appropriate time-step and the number of slosh modes necessary to produce results of sufficient accuracy. As mentioned earlier, Housner (4) and most subsequent researchers assumed that one slosh mode would be sufficient.

In order to test the method, a practical example was investigated and subjected to two accelerograms with rather different characteristics, namely the EW component of the 1940 El Centro record and the NS component of the 1977 Bucharest record.

A reservoir of the type illustrated by Fig. 1b was chosen with reservoir dimensions $R = 8.5m$ (27.9 ft), $D = 7.5m$ (24.6 ft), capacity = 1702 cu m (60,059 cu ft, or 449,200 gallons), giving a water mass of 1702 000 kg (116,553 slugs). Structural reservoir mass, including the contribution from column mass, was taken as 566 000 kg (38,758 slugs). The support system consisted of three concrete columns, each with an effective moment of inertia, allowing for cracking under lateral load, of $I = 0.481 m^4$ (55.7 ft⁴). The concrete modulus of elasticity was $E = 30 GPa$ (4.35×10^6 psi). With these data, the equivalent spring stiffness of the support system was $K = 154 MN/m$ (10,500 kips/ft). A viscous damping coefficient of $C = 0.536 MN/m/s$ (36.7 kip/ft/s) was adopted corresponding to approximately 2% critical damping in the columns.

Initial analyses were carried out to investigate the sensitivity of response to time-step and number of slosh modes using a two-second segment of the 1977 Bucharest record, starting at $T = 2.58$ s. This covers the time span of maximum ground acceleration. The short duration of the test segment was a consequence of the considerable computational effort, and hence computer cost, involved in running the program but was sufficient for comparative purposes.

Time Step - Figure 3 shows time-histories

TABLE 1 : SLOSH MODE PERIODS FOR
EXAMPLE RESERVOIRS

MODE	PERIOD (s)
1	4.48
2	2.53
3	2.00
4	1.71
5	1.52
10	1.06
15	.86

Fig. 3 INFLUENCE OF TIME-STEP ON RESPONSE TO BUCHAREST RECORD
(START AT 2.58 sec.)

Fig. 4 INFLUENCE OF NUMBER OF FREE SURFACE MODES ON
FREE SURFACE RESPONSE TO BUCHAREST RECORD.
(START AT 2.58sec)

of total shear force developed in the reservoir support system for time steps of $\Delta T = 0.1s$, $0.04s$ and $0.02s$. Although significant differences exist between the results for $\Delta T = 0.1s$ and the other records, the differences between $\Delta T = 0.02$ and 0.04 are less than 2% of peak response levels. It was felt that this degree of accuracy was adequate, and $\Delta T = 0.04s$ was used for all subsequent analyses.

Number of Slosh Modes - Response to the two-second segment of the Bucharest record for analyses using 1, 2, 5, 10 and 15 slosh modes is plotted in Fig. 4. The data plotted refer to fluctuations of fluid level at one end of the diameter parallel to the direction of seismic attack. It will be seen that although correlation between results for 10 and 15 modes is good, results using fewer modes give significant errors. In particular, the response for 1 or 2 modes bears little resemblance to the true response. On the basis of these results it was felt that a minimum of 10 slosh modes was necessary to obtain reasonably accurate predictions of time-histories of free-surface displacements. However, a similar analysis showed that 5 modes gave sufficient accuracy for the support system shear force.

The need for such a large number of modes was initially surprising to the authors. However, when the numerical values of the modal frequencies are investigated, the reasons become clear. Table 1 lists the periods of a selection of the first 15 modes. It will be noted that the modal periods are closely spaced, particularly for the higher modes, and that the first mode period, at 4.48s, is long in comparison with the period range of maximum energy typical of earthquakes. The 1977 Bucharest earthquake has unusually high energy levels in the longer period range, but it has a maximum spectral density of acceleration at a period of about 1.5s. Thus, it is clear from the modal periods listed in Table 1 that significant excitation of the higher modes can be expected, despite 'filtering' of the record by the structural response of the reservoir support system.

Examples of Response to Different Earthquakes

Figures 5 and 6 show response of the reservoir to the first 16s of the El Centro 1940 EW and Bucharest 1977 NS records, respectively. For comparative purposes, the corrected ground-displacement records of the two accelerograms have been included in the figures. It will be noted that the Bucharest record has a rather regular ground displacement, indicating strong periodicity at about 1.5s, while the El Centro record is less regular but has some rather long period components.

In both figures total shear force and free-surface displacement at the end of the diameter parallel to ground motion are plotted. It will be seen that in each case the support-system shear force is comprised largely of a

sinusoidal response of period 0.60s. However, despite the similarities of shear force response records, the free-surface displacement records are very different. Response to the El Centro record is characterised by a comparatively regular response at about 4.75s period, with higher mode contributions superimposed. The most significant higher mode contribution corresponds in period and phase to the shear force record and is clearly due to the forcing function provided by the sinusoidal response of the support-system. The period of 4.75s is close to the first slosh mode of 4.48 sec.

The Bucharest free-surface response, shown in Fig. 6c, displays quite different behaviour. For the first 8s, the free surface displacement record is confused, but subsequently it settles down to a response with a period of about 2.5s. From Table 1 it will be seen that this corresponds closely to response in the 2nd mode.

As would be expected for a system dominated by long-period response, there is a strong correlation between free-surface displacement and ground displacement, particularly in the earlier part of the record.

It is instructive to compare these results with values predicted by Housner's simplified two-mass system. Using the approach outlined in ref. 5, the water mass contributing to the inertial mode is found to be $M_0 = 847\,000\text{ kg}$ (58,012 slugs) and the water mass responding in the first mode $M_1 = 591\,500\text{ kg}$ (40,497 slugs). The stiffness of the equivalent spring for the first slosh mode is $k_1 = 1.13\text{ MN/m}$ (77.5 kips/ft). With these figures, the natural period of the inertial mode and first slosh mode are calculated to be 0.61s and 4.54s respectively. These agree well with the values of 0.60s and 4.48s, respectively, calculated by the method presented herein.

Using these values and response spectra for the relevant accelerograms, maximum shear forces of 9.5 MN and 7.4 MN are predicted for the El Centro and Bucharest records, respectively, which again compare well with the peak values from Figs. 5 and 6 of 10.0 MN and 7.05 MN, respectively. Maximum free surface displacements predicted by Housner's method are 0.72 and 0.45 m for El Centro and Bucharest respectively, compared with 0.710 m and 0.708 m for the more accurate approach presented in this paper. It appears that the apparently close agreement for the El Centro record is fortuitous, since, as evidenced by Fig. 4, the general shape of the displacement response for a single slosh-mode, as considered by the Housner model, bears little resemblance to the results from a multi-mode approach.

CONCLUSIONS:

A method has been presented, based on the classical approximations of linearized, inviscid wave mechanics, that enables time-histories of response of elevated cylindrical water reservoirs to seismic attack to be calculated for specified accelerograms.

Fig. 5 RESPONSE TO 16 SEC. OF EL CENTRO, 1940, E-W ACCELEROGRAM

Fig. 6 RESPONSE TO 16 SEC. OF 1977 BUCHAREST N-S ACCELEROGRAM

It was found that several slosh modes needed to be included in the analysis in order to obtain a sufficiently accurate prediction of seismic response of the tank. In particular, at least 10 modes were needed to predict the free-surface displacement response to specific earthquake records. These findings are in conflict with the assumptions by earlier researchers that one slosh mode would be adequate.

Comparison of results for two accelerograms with predictions of peak response by Housner's simplified two-mass model shows that Housner's method provided good predictions of the period of the inertial and first slosh modes, and predicted peak shear force in the reservoir support-system with adequate accuracy for design purposes. However, the violation of basic principles of fluid dynamics inherent in the development of Housner's approach leads the authors to suggest caution in applying this model to applications significantly different from those previously verified by experimental results, or by comparison with more exact approaches, such as that developed in this paper. In particular, where prediction of free-surface displacements is required, a multi-mode model based on sound fluid mechanics principles should be adopted.

ACKNOWLEDGEMENT:

The authors would like to thank Dr J.B. Berrill for furnishing the seismic records that were used for the calculations.

APPENDIX I - REFERENCES:

1. Aslam, M., Godden, W.G. and Scalise, D.T. "Earthquake Sloshing in Annular and Cylindrical Tanks", Journal of the Engineering Mechanics Division, ASCE, Vol. 105, No. EM3, Proc. Paper 14643, June, 1979, pp. 371-389.
2. Clough, D.P. "Experimental Evaluation of Seismic Design Methods for Broad Cylindrical Tanks", Report No. UCB/EERC-77/10, Earthquake Eng. Res. Center, Univ. of California, Berkeley, 1977, 273 pp.
3. Hildebrand, F.B. "Methods of Applied Mathematics", second edition, Prentice-Hall, Englewood Cliffs, 1965, p.264.
4. Housner, G.W. "Dynamic Pressures on Accelerated Fluid Containers", Bull. Seismological Soc. of America, Vol. 47, 1957, pp.15-35.
5. Housner, G.W. "Nuclear Reactors and Earthquakes, Appendix F", Atomic Energy Commission Rept. No. TID-7024, 1963, pp.367-389.
6. Hunt, B. and Priestley, M.J.N. "Seismic Water Waves in a Storage Tank", Bull. Seismological Soc. of America, Vol. 68, No. 2, April, 1978, pp.487-499.
7. Jacobsen, L.S., "Impulsive Hydrodynamics of Fluid Inside a Cylindrical Tank and of Fluid Surrounding a Cylindrical Pier", Bull. Seismological Soc. of America, Vol. 39, 1949, pp. 189-204.
8. Jennings, P.C. (editor). "Engineering Features of the San Fernando Earthquake of Feb. 9, 1971" Rept. No. EERL 71-02 of the Earthquake Eng. Res. Lab., California Inst. of Technology, Pasadena, 1971.
9. Rinne, J.E. "Oil Storage Tanks, The Prince William Sound, Alaska, Earthquake of 1964 and Aftershocks", Vol. 2, Part A, U.S. Dept. of Commerce, 1967.
10. Steinbrugge, K.V. and Flores, R. "Engineering Report on the Chilean Earthquake of May 1960 - A Structural Engineering Viewpoint", Bull. of Seismological Soc. of America, Vol. 53, No. 2, 1963.

APPENDIX II - NOTATION

- A_n = dimensionless coefficient defined by Eq. 7;
- C = coefficient of viscous damping;
- D = undisturbed water depth;
- F = prescribed function of time;
- F_w = water force upon tank;
- G = function defined by Eq. 14;
- g = gravitational constant;
- H = function defined by Eq. 16;
- K = spring constant;
- M_r = reservoir mass;
- M_w = water mass;
- R = tank radius;
- S_i, T_i = coefficients defined by Eqs 19-20;
- t = time;
- U, V = coefficients defined by Eqs. 21-22;
- X = horizontal ground displacement;
- x = horizontal tank displacement;
- β_n = coefficient defined by Eq.4;
- λ_n = nth root of the first derivative of the first-order Bessel function of the first kind; and
- ξ, τ = dummy integration variables.