A CONSIDERATION OF THE TORSIONAL RESPONSE OF BUILDING FRAMES

A.Rutenberg*

ABSTRACT

History of elastic static procedures for the seismic analysis of torsionally unbalanced building structures is briefly reviewed. It is suggested that the provisions of NZS 4203:1976, accounting for modal coupling, are based on inconsistent interpretation of results from well known two-degrees-of-freedom models. An alternative dynamic procedure is described which, while retaining the basic two-dimensional features of NZS 4203:1976 torsional provisions, is equivalent to three-dimensional modal spectral analysis. The procedure also results in a substantial simplification of the analysis compared with standard dynamic computer techniques now available to the structural engineer.

1. INTRODUCTION

The seismic torsional provisions of the New Zealand loading code[1] are intended to cover two major sources of rotational effects: (1) torsional ground motion, (2) coupling of lateral and torsional modes of response resulting from either designed or accidental asymmetry in the structure or its mass distribution. It is recognized, although only approximately accounted for, that the two lateral components of the ground motion as well as the torsional one may combine to amplify the structural response.

Housner and Outinen's pioneering paper on dynamic amplification[2] clearly indicated that the static method of analysis in which the inertia forces are applied statically at the mass centre may significantly underestimate the maximum forces in some of the planar assemblages comprising the structural system of a typical building. They also found that the amplification depends on the lateral to torsional frequency ratio. It thus appeared that the static methods were doomed as far as earthquake analysis of asymmetric structures was concerned. They were saved, however, two years later when Bustamante and Rosenblueth[3] introduced the concepts of dynamic eccentricity and dynamic amplification of static eccentricity. The dynamic eccentricity was defined as the ratio of the expected maximum torque and the expected maximum shear in a single storey 2-DoF (degrees-of-freedom) system. The maxima were taken as the RSS (square root of the sum of squares) of the modal spectral values. The dynamic amplification of eccentricity was defined as the ratio of this eccentricity and the static one, namely the distance between the mass centre CM and the centre of rigidity CR (Fig. 1). Their work as well as later studies by Skinner et al.[4], Penzien and Chopra[5], Elorduy and Rosenblueth[6,7] and Penzien[8] showed that considerable amplification of torque was to be expected for closely spaced lateral and torsional frequencies and low damping ratios, and that it was always conservative to ignore torsion when computing the total base shear. These conclusions have been found to hold for several smooth response spectra[9,10], and also for the more general three-dimensional case[9].

Although writers of building codes were quick to adopt the dynamic amplification concept by stipulating factored static eccentricities for the static earthquake analysis of torsionally unbalanced buildings, it soon became apparent that certain problems could not be so easily overcome. Monosymmetric buildings (2-DoF) vibrate in two coupled modes but only one dynamic eccentricity was prescribed. It followed that unless substantial underestimates were to be tolerated on assemblages located on the side of CR away from CM (Fig. 1), i.e. the "stiffer" side, two eccentricities would have to be specified. The problem was further complicated by the need to allow, again in a simplified manner, for the effects of accidental eccentricity due to earthquake input in two perpendicular lateral directions, as well as for the torsional input and for inaccuracies involved in estimating the relative stiffness of assemblages. Two formulae for the dynamic eccentricity e_d were thus prescribed, one for on either side of the rigidity centre and each consisting of two parts: the first, apparently, allowed for the effect of modal coupling and the second for all the imponderables[7,11].

Indeed, it is now becoming apparent that the effect of torsional coupling on the various structural assemblages cannot be described solely by the two constants provided by the codes, or by the more accurate eccentricity amplification curves given in the literature[6,7,9,10], because the displacement of a given assemblage depends on the combination of the two modes of vibration <u>at its particular location.</u> These constants, of course, are completely defined by the static eccentricity and the frequency ratio, but the relation is not linear[12,13].

Last but not least is the fact that

* Senior Lecturer in Civil Engineering, Technion - Israel Institute of Technology, Haifa Israel, recently on leave at the University of Canterbury.

static formulae, like many other code provisions, tend to have a life of their own. Since the seismic loadings evaluated by means of the response spectrum technique (3-dimensional dynamic analysis) usually result in providing RSS estimates of inter-storey shear force and torque envelopes (e.g. [3,8,9,14]), it was only to be expected that the response of a particular frame or assemblage would be conventionally computed, as in static analysis, by considering the more unfavourable effect of the RSS shear and the RSS torque combination[15], i.e. as two separate loading cases. This procedure, which is inherently incorrect, is much cheaper in terms of computer outlay than the correct one which requires the RSS evaluation of all the relevant member actions in every frame. However, it has been found to be quite conservative[12]. This erroneous interpretation of results is by no means general as readers familiar with standard three-dimensional computer programs, such as TABS[16], well know.

In view of the difficulties encountered with the static methods of analysis for torsional coupling, it is doubtful whether their application should be encouraged, in particular, when sufficiently straight forward alternatives are available. The purpose of this paper is to demonstrate the limitations of the static approach with respect to modal coupling and to show that at least for a class of torsionally unbalanced buildings, i.e. regular structures with a small or large degree of eccentricity, the NZS 4203:1976[1] two-dimensional semi-dynamic procedure can be suitably modified to conform better with analytical results.

As in earlier studies on seismic torsional response, only linear elastic systems are considered. In current engineering practice the actions on the assemblages resulting from linear dynamic analysis are used as a basis for the capacity design of the structure. Since it is accepted that design for strength should be related to the probable elastic load distribution as affected by torsion[17], it follows that reasonably consistent set of assemblage actions should be aimed at.

First the response of a single storey asymmetric building to random base excitation [10] is used to show how the expected responses of assemblages located at various distances from the centre of rigidity compare with those computed by static approaches. A simple two-dimensional dynamic analysis procedure[9,18] which incorporates the basic features of NZS 4203 torsional provisions is then adapted to evaluate the earthquake response of planar assemblages of regular asymmetric multistorey building structures.

2. TWO DEGREES OF FREEDOM SYSTEMS

The system studied is an idealized single storey structure (Fig. 1) consisting of a floor with a total mass m, rigid in its own plane and supported laterally by several massless planar assemblages (e.g. shear walls or frames). For simplicity of presentation one axis of symmetry is assumed, so that only 2-DoF are considered: lateral displacement in the y direction of the mass centre CM relative to the ground, and a rotation θ about a vertical axis through CM, but the technique is equally applicable to the more general case of no symmetry. The offset of CM from the centre of rigidity CR is the static eccentricity e at the floor. The earthquake acceleration $\ddot{U}_g$ is assumed to act in the y direction only, and the rotational component of excitation is ignored. The equations of motion and the analysis procedure for this system[3,9,18] are given in Appendix A.

The effects of lateral-torsional coupling on the response of the system are well known, and graphical presentations showing the amplification of static eccentricity and the associated reduction in the lateral force response for flat and hyperbolic response spectra are readily available[6,7,9]. Similar graphs were prepared also for white noise base excitation[10] and are shown in Fig. 2. A comparison of the graphs drawn for the above three spectra indicated no striking differences unless the torsional rigidity is very low.

With such graphs it is possible to evaluate the RSS shear force and the RSS torque acting on the building as a whole, namely:

$$\text{RSS (V)} = V_o . SR = K_y \sqrt{y_1^2 + y_2^2} \quad (1)$$

$$\text{RSS (T)} = V_o . e . TR = K_{\theta o} \sqrt{\theta_1^2 + \theta_2^2} \quad (2)$$

in which V_o = the shear acting on a symmetrical structure with identical mass m and lateral stiffness K_y, SR and TR are respectively the shear and torque ratios as given in Fig. 2, $K_{\theta o}$ = the torsional rigidity defined at the centre of rigidity, y_k and θ_k (k = 1,2) are respectively the modal lateral and torsional displacements under the assumed earthquake excitation. In the absence of eccentricity the shear force acting on each frame is proportional to its stiffness. In eccentric structures, however, some problems arise. The static methods compute the forces acting on any given frame for two separate loading cases, namely the RSS shear and the RSS torque and then combine them in the usual strength of materials fashion. Note that in this method only the combination that produces the maximum stress is used. However, for an asymmetric structure the shear force and torque cannot be treated as two separate loading cases, since each one comprises the effects of the two vibration modes of the system. Fig. 3 compares the two procedures, and it is seen that in order to obtain the expected maximum response of a frame located at a distance a from the mass centre, the combined, i.e. RSS, effect of the two modes at this particular location should be considered.

At this point it is useful to recall the particular case shown in Fig. 4 studied by Elorduy and Rosenblueth in their well known 1968 paper[6], where the original dynamic eccentricity curves were presented. The special feature of this model, not to be found in a typical building structure, is the complete separation between the lateral load (or shear) resisting frame and the frames resisting torsion. For this model the static procedure is also correct since the shear force acts only on the frame in the y direction and the torque acts only on the two frames in the x direction, so that the RSS shear and the RSS torque do represent the expected maxima for each frame. In a typical structure a frame which is not perpendicular to the direction of excitation is affected by both components, and therefore, the different phases between rotation and translation within each mode are lost in the static procedure (Fig. 3).

The maximum lateral displacements computed by the static methods are compared with their "exact" dynamic counterparts in Fig. 5. The results refer to a single storey structure with a torsional to lateral frequency ratio $\Omega = 1.0$ (or $\Omega_o = 0.98$) leading to a near maximum amplification of eccentricity (Fig. 2), and an eccentricity ratio $e^* = e/\rho = 0.2$ (ρ = mass radius of gyration). This eccentricity ratio is approximately 0.06 and 0.08 of the plan dimension b perpendicular to the direction of excitation (Fig. 1) for long and square buildings respectively.

In Fig. 5 the maximum lateral displacement ratio (relative to the symmetrical case, i.e. y/y_o) is plotted against the non-dimensional frame location along the x-axis. A band rather than a curve is depicted for the results of the dynamic analysis to show the limited effect in this case of the choice of a particular spectrum on the maximum displacement. The three type of spectra banded are shown in the inset of Fig. 5. A somewhat wider band will be obtained with increasing eccentricity ratio. These observations are not valid for spiked spectra as is evident from Reference 13, in which some results based on the El Centro spectrum are presented. Note that the displacement of the rigidity centre is much smaller than for the symmetric case (77 percent), as can also be seen from Fig. 2, and so is the total design shear capacity if most of the shear resisting assemblages are located near CR. However, when the stiffness is distributed more evenly, the total shear capacity will be affected to a lesser extent.

The effect of varying the torsional to lateral frequency ratio Ω_o may be seen from the "upper bound" curve shown in Fig. 6 for the white noise acceleration spectrum ($S_a \sim T^{-\frac{1}{2}}$). On the side of CR away from CM this curve represents a frequency ratio of about 0.7, this being equivalent to $K_{\theta o} \cong 0.5\rho^2 K_{yo}$. Lower frequency ratios lead to much larger displacements with increasing distance from CR, but practical designs are unlikely to have these features. For higher frequency ratios, the response remains below the "upper bound" curve.

For the static analysis the New Zealand loading code[1] specifies two design eccentricities, namely:

$$e_{d1} = 1.7e - e^2/b + 0.1b \tag{3}$$

$$e_{d2} = e - 0.1b \tag{4}$$

whichever is the less favourable for the frame under consideration. These formulae are similar to those given by other earthquake codes with the exception of the second term on the right hand side of Eq. (3), which is designed to account for the reduction in torsional amplification with increasing eccentricity (Fig. 2). It appears that the two equations with the 0.1b terms removed were intended to provide bounding curves for the torsional-lateral couplings effect[7,11]. The displacements computed on the basis of these formulae result in the two sets of full straight lines in Fig. 5 and Fig. 6, originating from $e^* = 0.2$ at $y/y_o = 1$, each for long ($b \cong 3.4\rho$) and square ($b \cong 2.45\rho$) buildings respectively. The effect of adding the 0.1b term is shown in two more sets of lines above the former. The broken lines denoted "static" result when the theoretical magnification factors (Fig. 2) rather than the code values are used. It is thus seen that the forces predicted by the Code (excluding the effect of 0.1b) may substantially overestimate the actions on the assemblages located on the "flexible" side of the structure and may underestimate those on the "stiffer" side. However, the effect of the 0.1b term is quite substantial in this case. The so called static method is always conservative, although, as expected, it overestimates the actions on the "stiffer" side to a larger extent than those on the "flexible" side. It can be seen from Fig. 2b that with increasing eccentricity the lateral forces computed on the basis of the Code become increasingly conservative. In fact it may be argued that the absence of gross underestimation of assemblage actions on the "stiffer" side of the structure is rather more the result of overestimating the lateral forces than the accuracy of the torsional formula per se.

In summary, NZS 4203:1976 does not predict consistently the probable elastic load distribution among the assemblages as induced by torsional effects, and large discrepancies between these results and those obtained from modal analysis are likely to occur. This is mainly due to inconsistent interpretation of results from 2-DoF models.

In view of the limitations of the present static approaches, it appears that a sufficiently simple modal procedure may become more attractive, and a brief description thereof is given in the following section.

3. RSS PROCEDURE TO EVALUATE ASSEMBLAGE FORCES

The elastic seismic response of torsionally coupled single storey structures has been well documented[2-9,18,21]. However, most reports are not concerned with the evaluation of assemblage forces once the RSS overall responses have been computed, presumably because this appears to be straight forward. Although the procedure is indeed quite simple, it has been shown that a different but inconsistent approach has been followed. To fix ideas, the computational sequence leading to the "correct" assemblage forces is now outlined. For clarity of presentation, the detailed algebraic derivation leading to the forces response of 2-DoF systems, easily available elsewhere[4,6,9,18] is presented in the Appendix A.

(1) Compute the 2-DoF mode shapes for the mono-symmetric structure (for a completely asymmetric one, compute three mode shapes). The procedure is described in Appendix A.

(2) Compute the modal lateral and torsional

displacements for the specified spectrum (Fig. 3):

$$y_k = S_{dk} \Gamma_k \psi_{yk} \qquad k = 1,2 \qquad (5)$$
$$\rho\theta_k = S_{dk} \Gamma_k \psi_{\theta k}$$

in which ψ_{yk}, $\psi_{\theta k}$ = the lateral and torsional mode shapes, S_{dk} = the spectral displacement associated with the natural frequency ω_k, and Γ_k = the k-th modal participation factor. Expressions for ψ and Γ are given in Appendix A.

(3) Compute the modal displacement response of the assemblage located at a distance a from the mass centre (Fig. 3):

$$y_{ak} = y_k + a\theta_k \qquad (6)$$

This, of course, is identical to:

$$y_{ak} = S_{dk} \Gamma_k (\psi_{yk} + \frac{a}{\rho} \psi_{\theta k}) = S_{dk} \Gamma_k \psi_{ak} \qquad (7)$$

where ψ_{ak} is the modal ordinate at a.

(4) Evaluate the maximum response by using the appropriate RSS formulae (Eq. (A10) or (A11)) depending on the separation of ω_1 and ω_2, namely

$$y_a = \mathrm{RSS}\ (y_{ak}) \qquad (8)$$

The shear force on the assemblage is then given by:

$$s_a = k_{ya}\, y_a \qquad (9)$$

where k_{ya} is the lateral stiffness of the assemblage located at a.

The application of the proposed technique to a class of multistorey building structures is discussed in the following section.

4. MULTISTOREY BUILDINGS

It is well known that in irregular structures as well as in some regular ones (e.g. regular walls and frames) a single vertical axis of rigidity cannot as a rule be defined. Even in a shear building, where the concept of rigidity centre is meaningful separately for every storey, the locus of these centres will not necessarily be a vertical one, nor will the principal axes of rigidity (axes of bending) for all storeys be identically oriented. This fact imposes, of course, certain limitations on the applicability of the two or three DoF models to the dynamic analysis of multistorey building structures.

It has been shown [9,14,18] that for a relatively wide class of asymmetric structures it is possible to simplify the modal spectral analysis considerably by taking advantage of their special geometric features. Strictly, the simplified procedure is applicable to asymmetric buildings with the mass centres of the floors being located along one vertical axis, and having a single family of framing systems comprising several planar assemblages with similar stiffness properties and a common variation thereof along the height of the building. However, it has been shown [9,18] that results with an accuracy sufficient for engineering purposes can be obtained when these conditions are only approximately satisfied.

A similar procedure may also be applied to structural systems having two families of resisting element with their axes of rigidity not necessarily co-linear. The results are again approximate, although it has been found that the approximation is usually of sufficient accuracy [18].

Briefly, the procedure consists of uncoupling 2N-DoF systems (or 3N-DoF for the general case of no symmetry) into N 2-DoF (or 3-DoF) systems which are linearly related to two N-DoF systems, N being the number of storeys. When the basic assumptions stated in the foregoing paragraph are practically satisfied, only one N-DoF system needs to be solved. In other words, the response of a torsionally coupled regular building can be expressed in terms of the response of one or at most three torsionally uncoupled systems, i.e. a symmetrical structure with its lateral stiffness properties identical to their counterparts in the actual structure, and associated torsionally unbalanced single storey structures of the type described in the previous section and in Appendix A. A short description of the technique is given in Appendix B.

It is seen that a particular feature of the class of structures under consideration is that the evaluation of natural frequencies and mode shapes requires only a two-dimensional (plane frame) rather than a three-dimensional (space frame) analysis. The proposed technique is thus similar in its major features to the procedure recommended by NZS 4203:1976 (Amendment No. 2) for the two-dimensional dynamic analysis of regular asymmetric building structures. It thus retains the Code's computational advantages since only plane frame computer programs are required. The basic difference lies, however, in the treatment of eccentricity. Whereas the Code uses Eqs. (3) and (4), the proposed method applies spectral analysis to compute the probable distribution of torsional effects on the various assemblages.

The sequence of steps in the analysis of response is thus similar to the one followed in section 3, with the exception that two sets of plane frame natural frequencies ω_{yj} and $\omega_{\theta j}$ as well as their respective mode shapes $\underline{\phi}_{yj}$ and $\underline{\phi}_{\theta j}$ must first be computed.

With the lower modal displacements of the assemblage (six for the case of one axis of symmetry) computed, the modal base shears, modal member axial forces and moments can be easily evaluated by using standard procedures. Finally, the probable maximum response is obtained by means of the appropriate RSS formula. Note, however, that in computing the response (e.g. y_a), by means of the RSS formula (step 4), Eq. (A11) becomes less tractable when sets of more than two closely spaced frequencies occur.

The computing sequence of member forces outlined in the foregoing paragraph is rather expensive in terms of computer outlay since all member actions have to be independently computed from their modal components. However, engineering practice usually accepts the responses represented by RSS storey shear forces as the equivalent loading where from member forces are derived by means of a standard structural analysis. Strictly, such a procedure does not produce the probable maximum member actions in the RSS sense, since they are computed on the basis of the RSS shear, rather than being computed independently. Yet such a procedure provides reasonable estimates of the maxima. It is also apparent that the more shear dependent the actions are the more accurate the accepted procedure becomes.

5. SUMMARY AND DISCUSSION

The limitations of the static methods for earthquake analysis of torsionally unbalanced building structures were discussed, and it was shown that some of the difficulties result from misinterpreting established procedures. A consistent procedure for the modal spectral analysis of torsionally unbalanced building structures has been adopted to compute the lateral forces on the assemblages. In its general features the technique is similar to the procedure recommended by NZS 4203:1976 for regular structures with large or small eccentricities.

For the type of structure to which the proposed procedure is applicable, a three-dimensional modal analysis can be easily carried out by means of two-dimensional programs. In terms of spectral modal analysis the procedure leads to a correct distribution of assemblage forces, whereas the semi-static code procedure may not. The procedure also results in a substantial simplification of the analysis compared with standard computer techniques now available to the structural engineer(16).

It may be reasonable to assume that asymmetry in the stiffness distribution does not appreciably affect the total energy stored by the structure, so it may appear that the strength distribution among the assemblages is not of utmost importance provided sufficient lateral capacity and some excess ductility are available. Therefore, migration of the centre of rotation resulting from uneven yielding may not be a major design concern. However, it is widely believed that when capacity design is not sufficiently related to the probable elastic force distribution among assemblages as induced by torsion, uneven rates of stiffness and strength degradation might lead to earlier failure of the weaker frames(17). Also, there is some evidence to the effect that the ductility factor method of analysis, i.e. an analysis in which equal elastic and plastic deformations are assumed to occur(19), may also be applicable to torsional problems (20). On the other hand it may be argued that underestimating the strength of frames on the "stiffer" side of the building relative to those on the "flexible" side, as Code based designs are likely to do in some cases, might, for the same reasons, reduce torsional effects by shifting the effective axis of rigidity in the direction of the mass axis.

It is not known to what extent an approach that advocates a probabilistically more consistent lateral force distribution is likely to lead to a much better seismic response. However, if agreement with probable elastic maxima is the criterion, static methods of torsional analysis are not sufficiently accurate when coupling effects are strong. At present, not much is known on the inelastic behaviour of torsionally unbalanced building structures and the subject deserves further study.

6. ACKNOWLEDGEMENTS

The author is indebted to Professor T. Paulay for his constructive criticism of the manuscript. This report was prepared while the author was on study leave at the University of Canterbury. The financial support provided by the University is gratefully acknowledged.

7. REFERENCES

1. Standards Association of New Zealand, NZ4203:1976 "Code of Practice for General Structural Design and Design Loadings for Buildings", Wellington, February 1976, 80 pp.
2. Housner, G. W. and Outinen, H., "The Effect of Torsional Oscillations on Earthquake Stresses", Bull. Seismological Society of America, Vol. 48, 1958, pp. 221-229.
3. Bustamante, J. I. and Rosenblueth, E., "Building Code Provisions on Torsional Oscillations", Proc. 2nd World Conf. Earthquake Engineering, Vol. 2, Japan, 1960, pp. 879-894.
4. Skinner, R. I., Skelton, D. W. C., and Laws, P. A., "Unbalanced Buildings and Buildings with Light Powers Under Earthquake Forces", Proc. 3rd World Conf. Earthquake Engineering, New Zealand, Vol. 2, 1965, pp. 586-602.
5. Penzien, J. and Chopra, A. K., "Earthquake Response of Appendages on a Multistorey Building", Proc. 3rd World Conf. Earthquake Engineering, New Zealand, Vol. 2, 1965, pp. 476-486.
6. Elorduy, J., Rosenblueth, E., "Torsiones Seismicas en Edificios de Un Piso", Report 164, Engineering Institute, National University of Mexico, Mexico D.F., April 1968, 32 pp.
7. Newmark, N. M. and Rosenblueth, E., "Fundamentals of Earthquake Engineering", Prentice Hall, Englewood Cliffs, 1970, 640 pp.
8. Penzien, J., "Earthquake Response of Irregularly Shaped Buildings", Proc. 4th World Conf. Earthquake Engineering, Chile, Vol. 2, 1969, pp. 75-88.
9. Kan, C. L. and Chopra, A. K., "Coupled Lateral Torsional Response of Building to Ground Motion", Report EERC 76-13, Earthquake Engineering Research Centre, Univ. of California, Berkeley, May 1976, 167 pp.
10. Rutenberg, A., "Response of Single Storey Asymmetric Buildings to a White Noise Base Excitation", Report in preparation.
11. Elms, D. G., "Seismic Torsional Effects on Buildings", Bull. N.Z. National Society for Earthquake Engineering, Vol. 9, No. 1, 1976, pp. 79-83.
12. Rutenberg, A., Hsu, T. I. and Tso, W.K., "Response Spectrum Techniques for Asymmetric Buildings", Earthquake Engineering & Structural Dynamics, Vol. 6, 1978, pp. 427-435.

13. Elms, D. G., "Seismic Torsional Coupling in Structures", Proc. 6th Australasian Conf. on the Mechanics of Structures & Materials, Christchurch, New Zealand, 1977, pp. 189-193.
14. Gisbon, R. F., Moody, M. L., and Ayre, R. S., "Response Spectrum Solution for Earthquake Analysis of Asymmetrical Multistoried Buildings", Bull. Seismological Society of America, Vol. 62, 1972, pp. 215-229.
15. Irwin, A. W., "Analysis of Tall Shear Wall Buildings Including In-Plane Floor Deformations", Build International, Vol. 8, 1975, pp. 43-56.
16. Wilson, E. L. and Dovey, H. H., "Three Dimensional Analysis of Building Systems - TABS", Earthquake Engineering Research Centre Report EERC 72-8, University of California, Berkeley, December 1972, 47 pp.
17. Poole, R. A., "Analysis for Torsion Employing Provisions of NZRS 4203: 1974", Bull. N.Z. Nat. Society for Earthquake Engineering, Vol. 10, No. 4, 1977, pp. 219-225.
18. Reinhorn, A., Rutenberg, A., and Gluck, J., "Dynamic Torsional Coupling in Asymmetric Building Structures", Building and Environment, Vol. 12, 1977, pp. 251-261.
19. Clough, R. W., and Penzien, J., "Dynamics of Structures", McGraw-Hill, Kogakusha, Tokyo, 1975, 634 pp.
20. Shiga, T. et al, "Torsional Response of Structures to Earthquake Motion", Proc. U.S. - Japan Seminar on Earthquake Engineering, Sendai, Japan, 1970, pp. 156-171.
21. Kosko, E., "The Frequency Spectrum of a Structural Member in Coupled Flexural Torsional Vibration", J. Sound and Vibration, Vol. 7, 1968, pp. 143-155.
22. Clough, R. W., "Earthquake Resistant Design of Tall Buildings", Proc. Symp. Tall Buildings, Planning Design and Construction, Nashville, Tenn. 1974. pp. 477-506.

8. NOTATION

a	= distance of assemblage from mass centre
a_o	= modal shape zero-crossing point from mass centre (Fig. 3)
b	= plan dimension of building perpendicular to direction of excitation
CM	= centre of mass
CR	= centre of rigidity
DoF	= degrees-of-freedom
e	= static eccentricity
e_d	= dynamic or design eccentricity
e^*	= e/ρ = eccentricity ratio
K_y	= lateral rigidity (along y-axis)
$K_\theta, K_{\theta o}$	= torsional rigidity defined at centre of mass or rigidity respectively
$\underset{\sim}{K}$	= stiffness matrix
m	= storey mass
$\underset{\sim}{M}$	= mass matrix
N	= number of storeys
RSS	= square root of sum of squares
s_a	= acceleration spectrum
s_d	= displacement spectrum
SR	= shear ratio
T	= torque or natural period of vibration
TR	= torque ratio
u	= uncoupled generalised coordinates (single storey structure)
$\ddot{U}_g$	= ground acceleration time history
V	= shear force
V_o	= shear force in the associated symmetrical system
x	= axis of symmetry
y	= lateral displacement or displacement vector
y_o	= lateral displacement of the related symmetric structure
γ	= planar mode shape coefficient
Γ	= modal participation factor
ε	= correction coefficient in RSS formula
η	= damping ratio
θ	= angle of rotation
ρ	= mass radius of gyration
ϕ	= mode shape vector along height of building
$\underset{-}{\phi}_o$	= mode shape vector along height of building for associated 2-dimensional system
ψ	= planar mode shape of torsionally uncoupled 2-DoF system
$\omega_{1,2}$	= circular frequencies of torsionally coupled system
ω_y	= uncoupled translational circular frequency
$\omega_\theta, \omega_{\theta o}$	= uncoupled torsional circular frequency defined at mass or rigidity centre respectively
Ω, Ω_o	= torsional to lateral uncoupled frequency ratio defined at mass or rigidity centre respectively

9. APPENDIX A - TWO DEGREES OF FREEDOM SYSTEMS

The two degrees of freedom for the system given in Fig. 1 are the lateral displacement y and a rotation θ about the vertical axis. The lateral and torsional rigidities K_y and K_θ of the 2-DoF system are obtained from the individual resisting members (assemblages, if they are not simple cantilvers) in the usual way, namely

$$K_y = \sum_i K_{iy} \tag{A1}$$

$$K_\theta = \sum_i K_{iy}\, x_i^2 + \sum_i K_{ix}\, y_i^2 + \sum_i K_{i\theta} = K_{\theta o} + K_y\, e^2$$

where x_i and y_i are the perpendicular distances to the mass centre and $K_{i\theta}$ is the torsional rigidity of an assemblage about its own axis, which for planar members may be neglected. The eccentricity e of the rigidity centre CR from the mass centre CM is given by

$$e = \frac{1}{K_y} \sum_i K_{iy}\, x_i \tag{A2}$$

Under earthquake excitation $\ddot{U}_g$ in the y direction, the equations of motion for the system assuming elastic behaviour read:

$$\omega_y^2 \begin{bmatrix} 1 & e^* \\ & \\ e^* & \Omega^2 \end{bmatrix} \begin{Bmatrix} y \\ \\ \rho\theta \end{Bmatrix} + \begin{Bmatrix} \ddot{y} \\ \\ \rho\ddot{\theta} \end{Bmatrix} = - \begin{Bmatrix} \ddot{U}_g \\ \\ 0 \end{Bmatrix} \qquad \text{(A3)}$$

in which m = storey mass, ρ = mass radius of gyration about CM, and

$$\omega_y = \sqrt{K_y/m}\ , \quad \omega_\theta = \sqrt{K_\theta/m\rho^2} \qquad \omega_{\theta o} = \sqrt{K_{\theta o}/m\rho^2}$$

$$e^* = e/\rho\ , \quad \Omega^2 = \omega_\theta^{\ 2}/\omega_y^{\ 2} = \Omega_o^{\ 2} + e^{*2}$$

Note that $\rho\theta$ rather than θ is used in order to render Eq. (A3) dimensionally compatible. Since orthogonal damping[19] is assumed the relevant term does not appear in Eq. (A3) but it will be defined in the two uncoupled modal equations that follow. The uncoupled frequencies ω_1 , ω_2 and the modal matrix ψ can be easily obtained from the associated eigenvalue problem of Eq. (A3) and are given by

$$\omega_{1,2}^{\ 2}/\omega_y^{\ 2} = \frac{1+\Omega^2}{2} \pm \sqrt{(1 - \Omega^2)^2/4 + e^{*2}} \qquad \text{(A4)}$$

and

$$\underset{\sim}{\psi} = \begin{bmatrix} \psi_{y1} & \psi_{y2} \\ & \\ \psi_{\theta 1} & \psi_{\theta 2} \end{bmatrix} = \frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} 1 & \gamma \\ & \\ -\gamma & 1 \end{bmatrix} \qquad \text{(A5)}$$

where

$$\gamma = (1 - \omega_1^{\ 2}/\omega_t^{\ 2})\ /e^*$$

Letting

$$\begin{Bmatrix} y \\ \\ \rho\theta \end{Bmatrix} = \underset{\sim}{\psi} \begin{Bmatrix} u_1 \\ \\ u_2 \end{Bmatrix} \qquad \text{(A6)}$$

premultiplying Eq. (A3) by the transpose of ψ and adding the damping terms, the following uncoupled equations are obtained:

$$\begin{aligned} \ddot{u}_1 + 2\eta_1\,\dot{u}_1 + \omega_1^{\ 2}\,u &= -\,\Gamma_1\,\ddot{U}_g \\ \ddot{u}_2 + 2\eta_2\,\dot{u}_2 + \omega_2^{\ 2}\,u &= -\,\Gamma_2\,\ddot{U}_g \end{aligned} \qquad \text{(A7)}$$

where the modal participation factors Γ_k (k = 1,2) are given by

$$\Gamma_1 = \frac{1}{\sqrt{1+\gamma^2}}\ ; \quad \Gamma_2 = \frac{\gamma}{\sqrt{1+\gamma^2}}$$

and η_1 and η_2 are the damping ratios. Using the harmonic relation between the spectral values we have $S_d = S_a/\omega^2$, where S_a is the specified acceleration spectrum. Once S_d and Γ are computed, it is possible to evaluate the modal displacement of any frame (at a distance a from CM) using the following expressions:

$$\psi_{ak} = \psi_{yk} + \frac{a}{\rho}\,\psi_{\theta k} \qquad \text{(A8)}$$

and

$$y_{ak} = S_{dk}\,\Gamma_k\,\psi_{ak} \qquad \text{(A9)}$$

The most probable response of a multi-degree of freedom system with well separated natural frequencies is given by the root square of the sum of squares (RSS) formula, so that

$$\text{RSS}\ (y_{ak}) = \sqrt{y_{a1}^{\ 2} + y_{a2}^{\ 2}} \qquad \text{(A10)}$$

When ω_1 and ω_2 are sufficiently close, as is often the case with asymmetric buildings, the corrected RSS formula should be used[6,7]:

$$\text{RSS}\ (y_{ak}) = \sqrt{y_{a1}^{\ 2} + y_{a2}^{\ 2} + \frac{2y_{a1}\,y_{a2}}{1+\varepsilon^2}} \qquad \text{(A11)}$$

where for small values of η_k and $\eta_1 = \eta_2 = \eta$

$$\varepsilon \cong \frac{\omega_1 - \omega_2}{\eta\,(\omega_1 + \omega_2)} \qquad \text{(A12)}$$

Note that in Eq. (A11) y_{ak} is to be taken with the sign that its unit impulse function has when it attains its maximum numerical value.

Sometimes the zero crossing points of the mode shapes (a_{ok} in Fig. 3) are used to evaluate the modal displacements along the x-axis. The relevant expressions were first given by Kosko[21]:

$$a_{ok} = e/(1 - \omega_k^2/\omega_y^{\ 2}) \quad ; \quad k = 1,2 \qquad \text{(A13)}$$

The importance of this expression lies, however, in the light that it sheds on the lateral-torsional coupling problem. When $\Omega = 1$ it can easily be shown that a_{ok} is independent of e, i.e. the mode shapes are the same for large and small eccentricities. This fact led some engineers to believe that violent rotations might be expected even with small eccentricity provided the frequencies are sufficiently close. This however does not appear to be the case as can easily be verified by means of Eq. (A11): the two modes are practically in phase so that algebraic summation gives the same results.

10. APPENDIX B - MULTI DEGREE OF FREEDOM SYSTEM

The dynamic properties of a mono-symmetric torsionally unbalanced building structure may be obtained from the solution of the eigenvalue problem given by:

$$\left(\begin{bmatrix} \underset{\sim}{K}_y & e^*\underset{\sim}{K}_y \\ & \\ e^*\underset{\sim}{K}_y & \frac{1}{\rho^2}\underset{\sim}{K}_\theta \end{bmatrix} - \omega^2 \begin{bmatrix} \underset{\sim}{M} & \underset{\sim}{O} \\ & \\ \underset{\sim}{O} & \underset{\sim}{M} \end{bmatrix} \right) \begin{Bmatrix} \theta_y \\ \\ \rho\,\Phi_\theta \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{O} \\ \\ \underset{\sim}{O} \end{Bmatrix} \qquad \text{(B1)}$$

in which $\underset{\sim}{K}_y$, $\underset{\sim}{K}_\theta$ = the lateral and torsional stiffness matrixes of order N (= number of storeys) with respect to CM, $\underset{\sim}{M}$ = diagonal mass matrix, ρ = mass radius of gyration and Φ_y, Φ_θ = lateral and torsional mode shape vectors. Note that only when $\underset{\sim}{K}_\theta$ is proportional to $\underset{\sim}{K}_y$ is the following procedure exact. Methods for assembling $\underset{\sim}{K}_y$ from the individual properties of the assemblages are well known (e.g. [22]) and will not be discussed here.

The uncoupled procedure which was described in some detail in References 9 and 18 will be now briefly outlined. Let ω_y, ω_θ and Φ_{yo}, $\Phi_{\theta o}$ be the natural frequencies and mode shapes respectively of the following two uncoupled N x N systems, then:

$$(\underset{\sim}{K}_y - \omega_y^{\ 2} \underset{\sim}{M}) \ \Phi_{yo} = 0 \tag{B2}$$

$$(\underset{\sim}{K} - \omega_\theta^{\ 2} \rho^2 \underset{\sim}{M}) \ \Phi_{\theta o} = 0 \tag{B3}$$

Note that each of these two systems is a problem of the type that must be solved when the dynamic two-dimensional procedure for asymmetric buildings of NZS 4203:1976 is followed. The natural frequencies of the actual system (Eq. (B1)) are given by

$$(\omega_{1,2}^{\ 2}/\omega_y^2)_j = (1 + \Omega_j^2) \ /2 \pm \sqrt{(1 - \Omega_j^2) \ ^2/_4 + e^{*2}} \tag{B4}$$

where

$$\Omega_j^2 = \omega^2 {}_{\theta j}/\omega_{yj}^2$$

which are identical to the 2-DoF expressions given in Appendix A. It can also easily be shown that the mode shapes of Eq. (B1) may be expressed by the N-DoF mode shapes factored by the 2-DoF modal ordinates, namely:

$$\Phi_{yjk} = \Phi_{yoj} \ \ \psi_{yjk} \qquad j = 1....N, \ k = 1,2 \tag{B5}$$

$$\rho \Phi_{\theta jk} = \Phi_{\theta oj} \ \psi_{\theta jk} \tag{B6}$$

where

$$\underset{\sim}{\psi}_j = \begin{bmatrix} \psi_{yj1} & \psi_{yj2} \\ \psi_{\theta j1} & \psi_{\theta j2} \end{bmatrix} = \frac{1}{\sqrt{1 + \gamma_j^2}} \begin{bmatrix} 1 & \gamma_j \\ -\gamma_j & 1 \end{bmatrix} \tag{B7}$$

and

$$\gamma_j = (1 - \omega_{j1}^{\ 2}/\omega_{yj}^{\ 2})/e^*$$

which are identical to Eq. (A5).

As has already been pointed out, in elastic analysis we are interested in the response of the assemblages (walls and frames) comprising the structural system rather than in the overall lateral or torsional forces acting on the whole building. Therefore for a given assemblage situated at a distance a along the x-axis from the mass centre we have:

$$\Phi_{ajk} = \Phi_{yjk} + \frac{a}{\rho} \Phi_{\theta jk} \ ; \quad j = 1....N, \quad k = 1,2 \tag{B8}$$

The modal displacement of the assemblage due to earthquake base acceleration $\ddot{U}_g$ are obtained from the expression:

$$Y_{ajk} = S_{djk} \ \Gamma_{jk} \ \Phi_{ajk} \ ; \quad j = 1....N, \quad k = 1,2 \tag{B9}$$

where S_{djk} is the spectral displacement associated with the natural frequency ω_{jk} and Γ_{jk} is the modal participation factor given by:

$$\begin{Bmatrix} \Gamma_{j1} \\ \Gamma_{j2} \end{Bmatrix} = \begin{bmatrix} \psi_j \end{bmatrix} \begin{bmatrix} \Phi_{yoj}^T & 0 \\ 0 & \Phi_{\theta oj}^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{M} & 0 \\ 0 & \underset{\sim}{M} \end{bmatrix} \begin{Bmatrix} \underline{1} \\ \underline{0} \end{Bmatrix} \tag{B10}$$

(2x1) (2x2) (2x2N) (2Nx2N) (2Nx1)

Paper received 15 December, 1978.

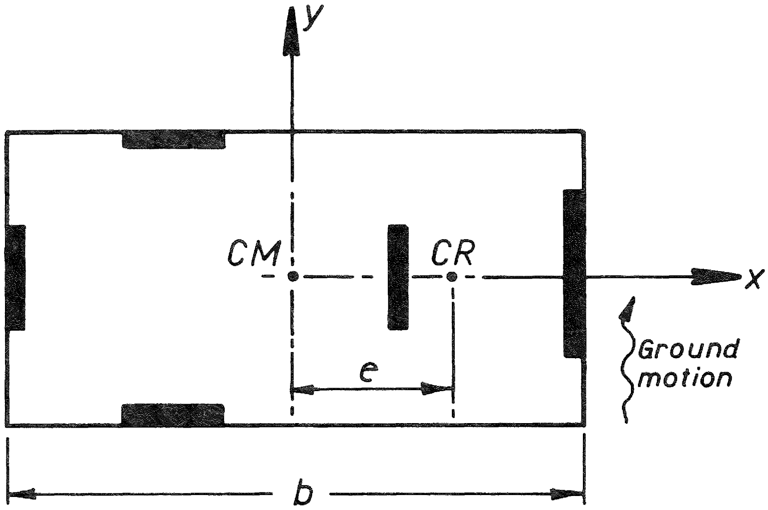


FIGURE 1: PLAN OF SINGLE STOREY STRUCTURAL MODEL

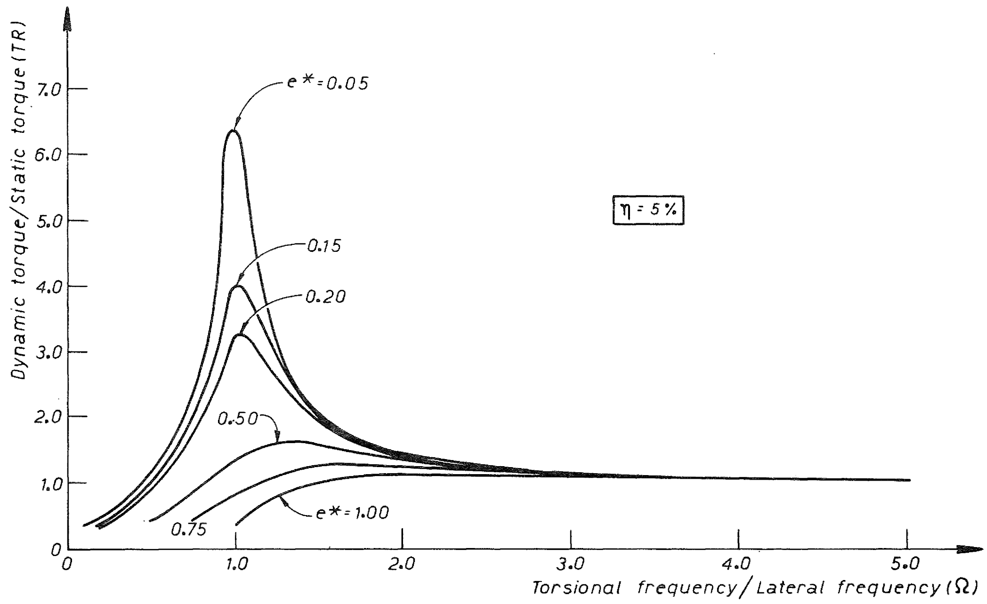


FIGURE 2: (a) TORSION AMPLIFICATION FACTOR

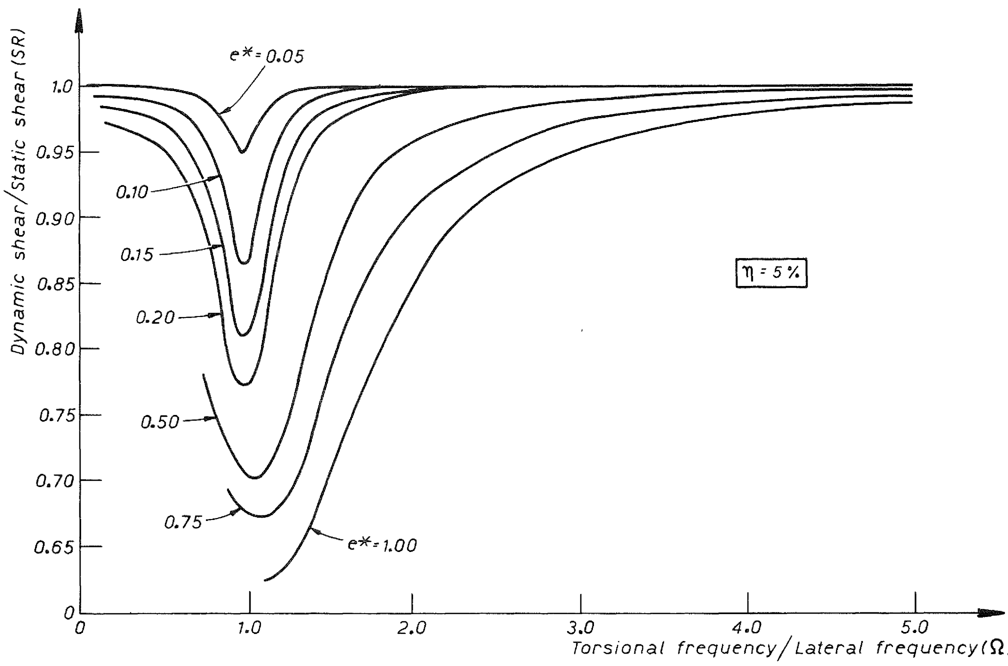
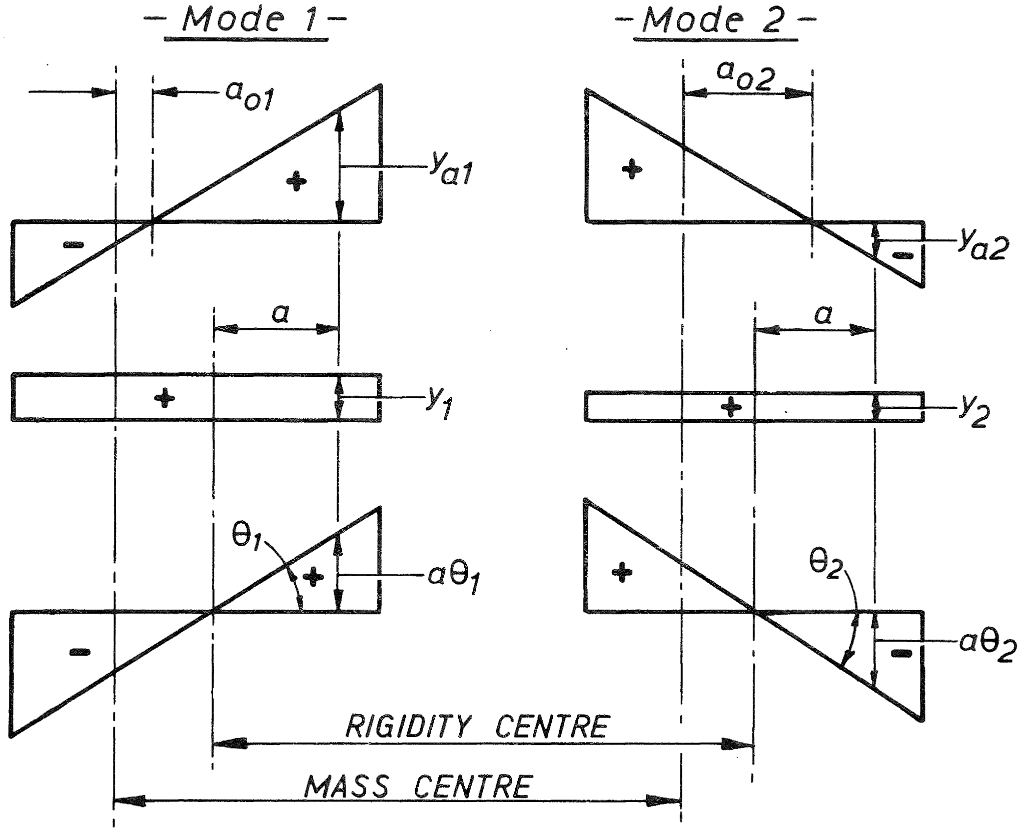


FIGURE 2: (b) RELATION BETWEEN DYNAMIC AND STATIC SHEAR



$$RSS(y_a) = \sqrt{y_{a1}^2 + y_{a2}^2}$$ Correct

$$RSS(y) = \sqrt{y_1^2 + y_2^2}$$
$$RSS(a\theta) = a\sqrt{\theta_1^2 + \theta_2^2}$$
$$RSS(y_a) = RSS(y) + RSS(a\theta)$$
Conventional (Incorrect)

FIGURE 3: CORRECT AND CONVENTIONAL RSS PROCEDURE FOR A GIVEN ASSEMBLAGE

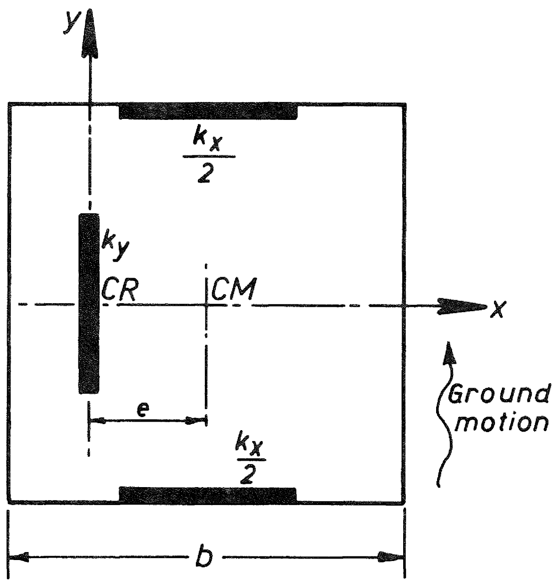


FIGURE 4: PLAN OF SINGLE STOREY STRUCTURE STUDIED BY ELORDUY AND ROSENBLUETH (6,7).

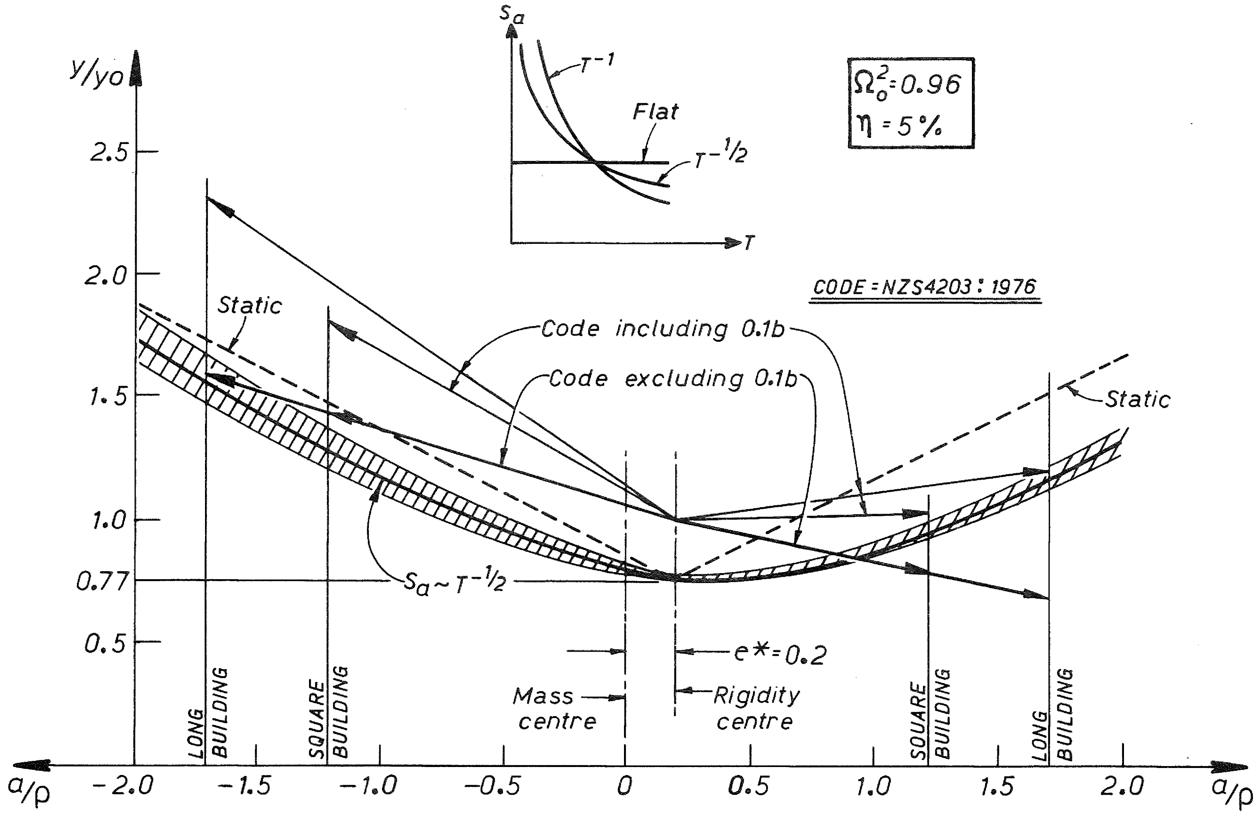


FIGURE 5: LATERAL DISPLACEMENT vs. FRAME LOCATION:
STATIC AND DYNAMIC ANALYSES

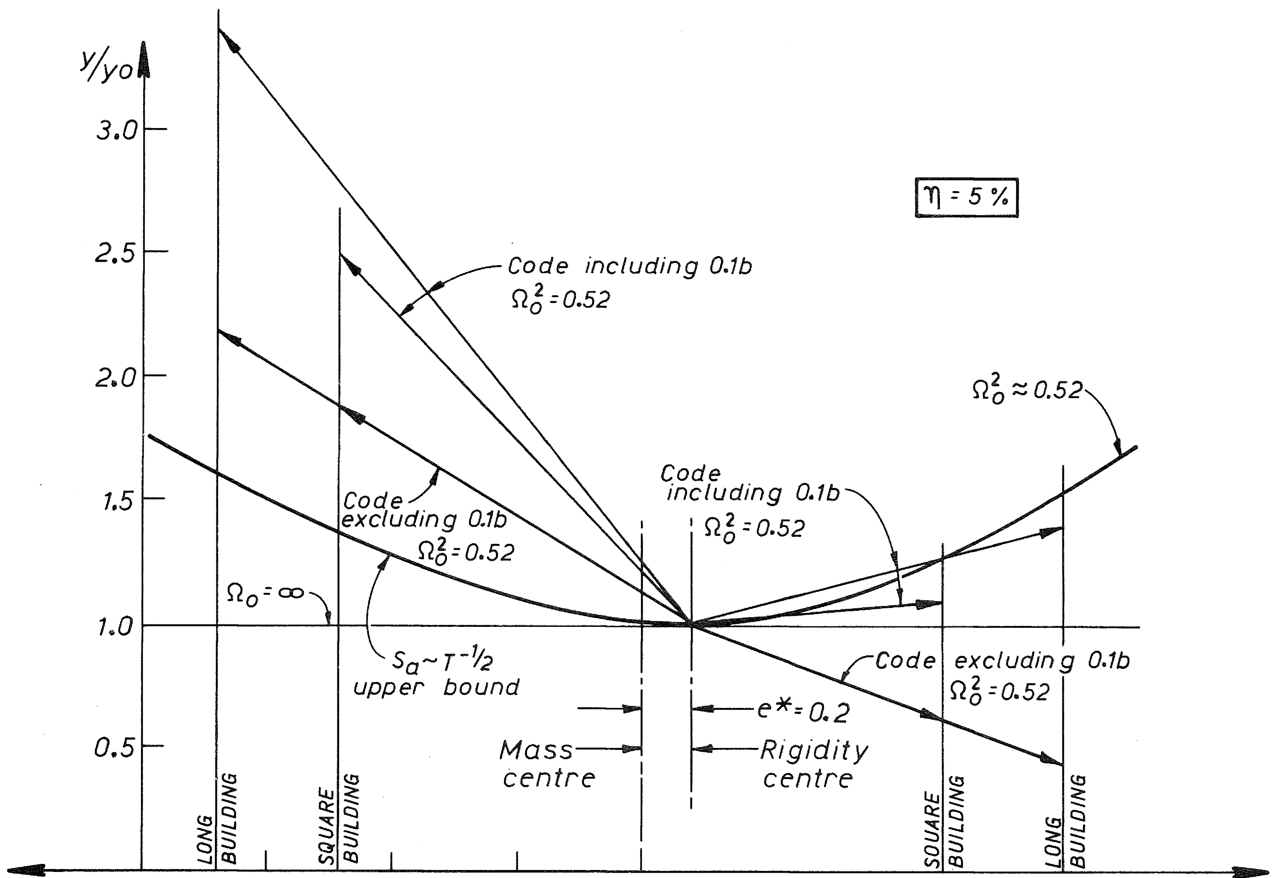


FIGURE 6: UPPER BOUNDS FOR LATERAL DISPLACEMENT OF FRAMES:
STATIC AND DYNAMIC ANALYSES WITH WHITE NOISE SPECTRUM:
e* = 0.2. (FOR THE CODE CASES CASES ONLY $\Omega_o 2$ = 0.52 WAS USED)